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Fashion Cycles: a Discrete Time Analysis

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Description of the map

2D piecewise linear discontinuous map

$F : I^2 \to I^2$, $I^2 = [0,1] \times [0,1]$,

$$F_1 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} (1-\delta)x \\ (1-\delta)y + \delta \end{pmatrix}, \quad (x,y) \in D_1;$$

$$\mathrm{F}_2 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} (1-\delta)x + \delta \\ (1-\delta)y \end{pmatrix}, \quad (x,y) \in D_2;$$

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where the **regions** are defined as

$$\begin{aligned} D_1 &= \{(x,y) : P^x < 0,\ P^y < 0\}; \quad D_2 = \{(x,y) : P^x > 0,\ P^y > 0\} \\ D_3 &= \{(x,y) : P^x < 0,\ P^y > 0\}, \quad D_4 = \{(x,y) : P^x > 0,\ P^y < 0\} \end{aligned}$$

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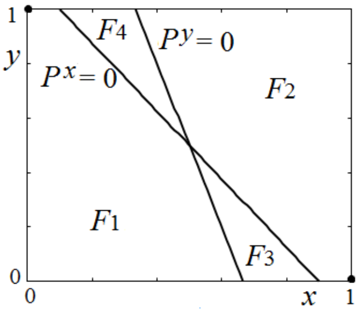

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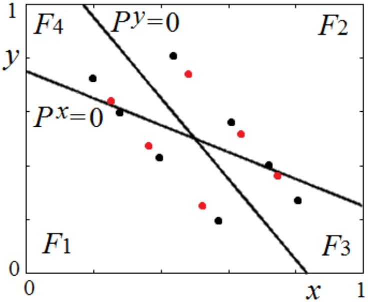

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Preliminaries

- Map F is **symmetric wrt** $(x, y) = (1/2, 1/2)$ denoted S.
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The **discontinuity lines**:
$y = -\frac{1}{m_x}x + \frac{1+m_x}{2m_x} \quad (C^x) \Leftrightarrow P^x = 0, \quad y = -m_y x + \frac{m_y+1}{2} \quad (C^y) \Leftrightarrow P^y = 0$
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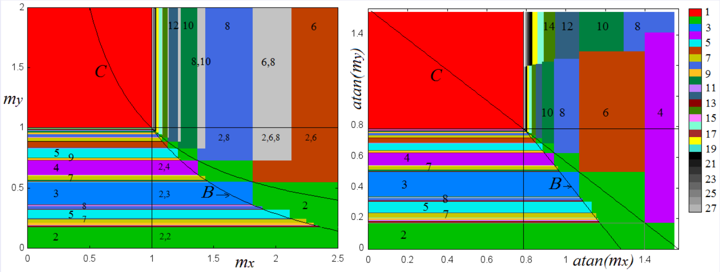
2D bif. diagram: period adding and period incrementing structures



Partitioning of the parameter space

Depending on $m_x m_y \gtrless 1$, $m_x \gtrless 1$, $m_y \gtrless 1$

• For $(m_x, m_y) \in R_I = \{m_x m_y > 1,\ m_x < 1\}$ (**Case I**) and $(m_x, m_y) \in R_{II} = \{m_x m_y < 1,\ m_y > 1\}$ (**Case II**) map F has two **attracting border fix. p-ts**, $(x, y) = (0, 1)$ and $(x, y) = (1, 0)$. Their basins are separated by C^x.

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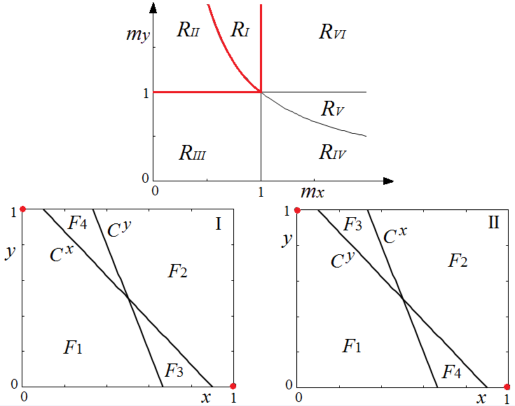

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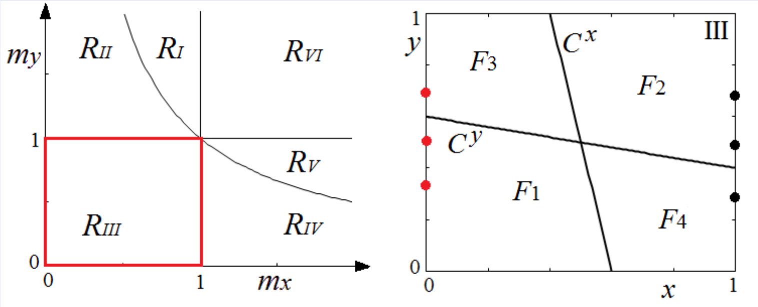

Attractors of F (Case III)

An n-**cycle** γ_n, $n \geq 2$, belonging to the left border I_0 of I^2, and an n-**cycle** γ_n' belonging to the right border I_1 of I^2. The basins of γ_n and γ_n' are separated by C^x.

Case III: two border n-cycles and period adding structure

The dynamics on I_0 (I_1) are governed by the **1D piecewise linear discontinuous map** g ($\overline{g}$, resp.) with the discontinuity point $c_{-1} = (m_y + 1)/2 > 1/2$ ($c'_{-1} = (1 - m_y)/2 = 1 - c_{-1} < 1/2$):

$$g : y \to g(y) = \begin{cases} g_L(y) = (1-\delta)y + \delta, & 0 \le y < c_{-1} \\ g_R(y) = (1-\delta)y, & c_{-1} < y \le 1 \end{cases}$$

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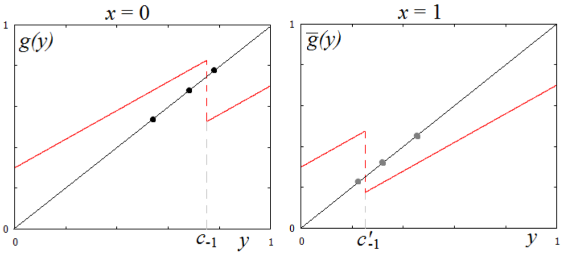

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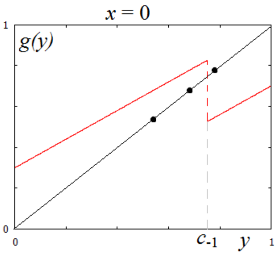

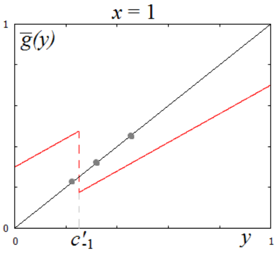

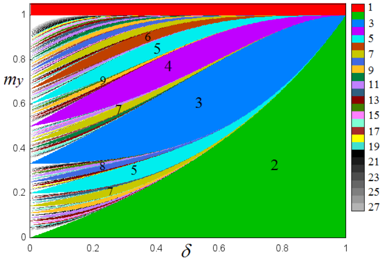

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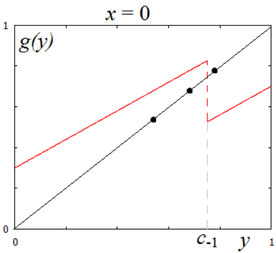
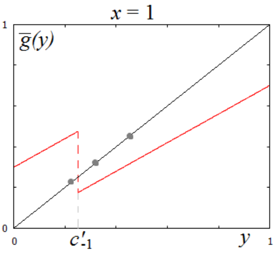
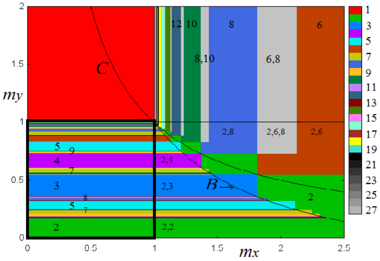


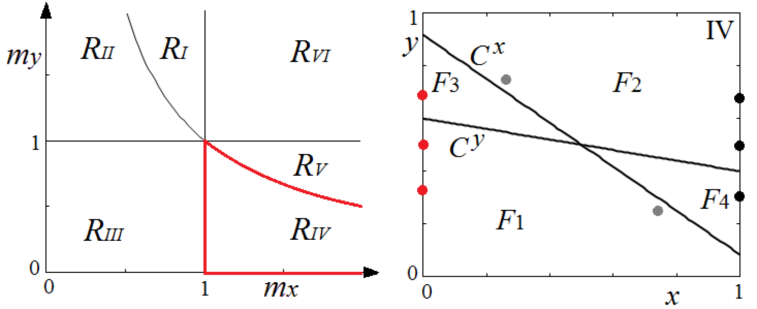




Partitioning of the parameter space

- For $(m_x, m_y) \in R_{IV} = \{m_x m_y < 1,\ m_x > 1\}$ (**Case IV**)

Case IV



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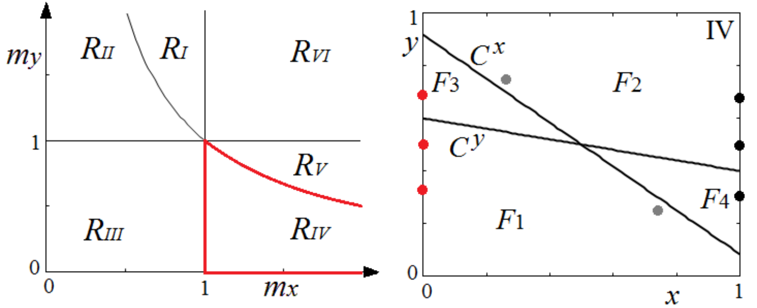
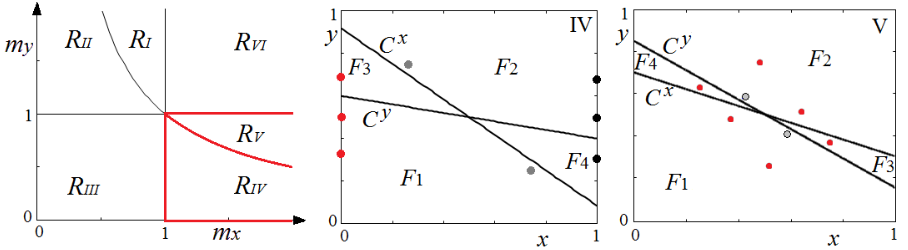


Attractors of map F (Case IV)

An interior 2**-cycle** Γ_2, which may coexist or not with **two attracting border** n**-cycles** $\gamma_n \in I_0$ and $\gamma'_n \in I_1$.

Cases IV and V: An interior 2-cycle



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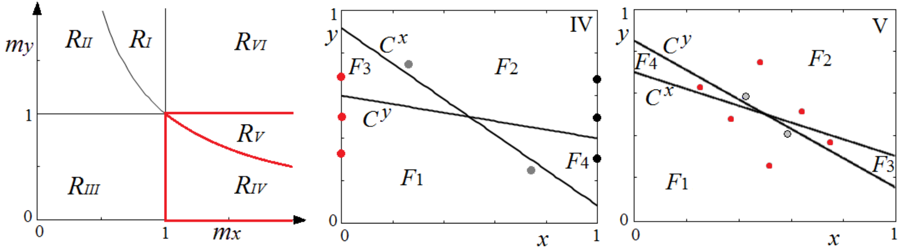


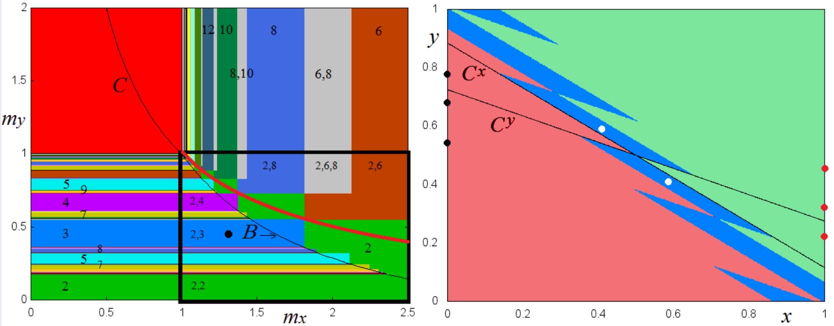
An interior 2**-cycle** $\Gamma_2 = \{p_0, p_1\}$, with

$$p_0 = \left(\frac{1}{2-\delta}, \frac{1-\delta}{2-\delta}\right) \in D_1, \quad p_1 = \left(\frac{1-\delta}{2-\delta}, \frac{1}{2-\delta}\right) \in D_2$$

exists for $m_x > 1$, $m_y < 1$ (i.e., $(m_x, m_y) \in R_{IV} \cup R_V$):
At $m_x = 1$ a BCB occurs at which $p_0 \in C^x$ (as well as $p_1 \in C^x$);
At $m_y = 1$ a BCB occurs at which $p_0 \in C^y$ (as well as $p_1 \in C^y$)

Interior 2-cycle

Case IV: Basins of attraction (below curve $B : m_y = \frac{1-\delta m_x}{m_x(1-\delta)}$)



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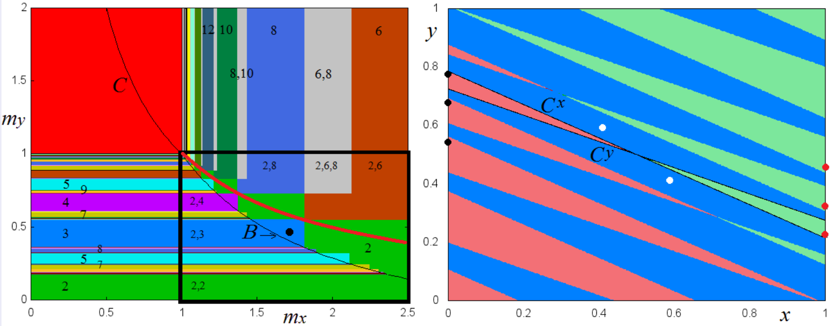

Partitioning of the parameter space

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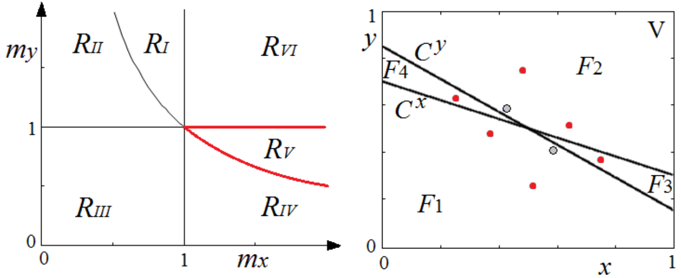

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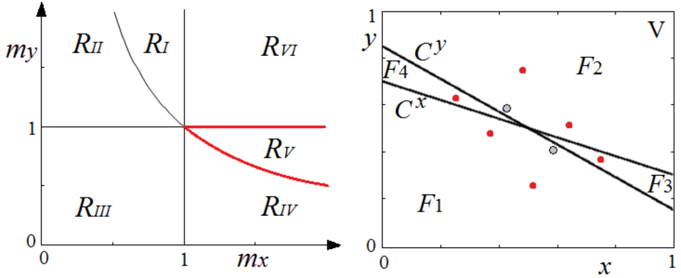
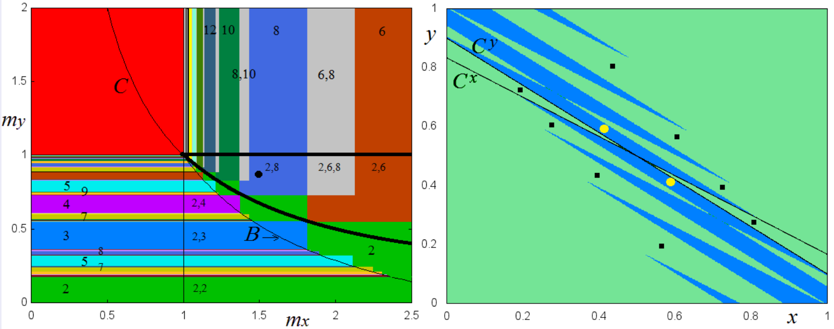


Attractors (Case V)

- Case IV $\Rightarrow$ Case V: the curves C^x and C^y are merging and switching their position.
- Borders I_0 and I_1 are no longer invariant, cycles γ_n and γ'_n no longer exist.
- 2-cycle Γ_2 may coexists or not with basic cycle Γ_{2n}, $n \geq 2$, or with Γ_{2n} and $\Gamma_{2(n+1)}$.

Case V: interior 2- and $2n$-cycles; incrementing structure



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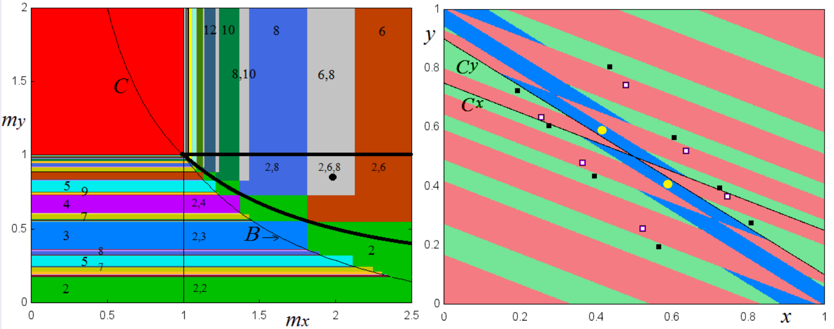

Partitioning of the parameter space

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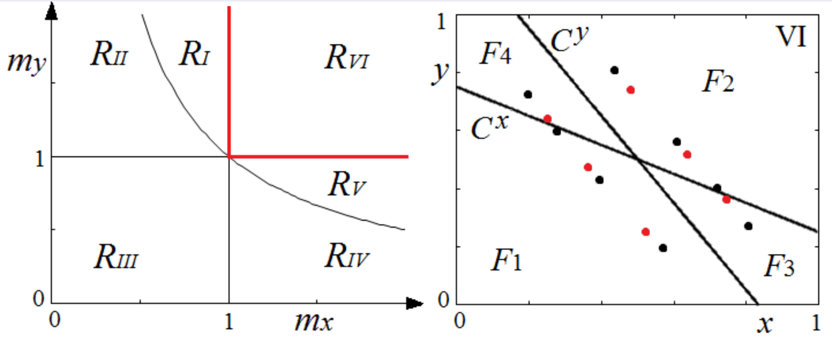

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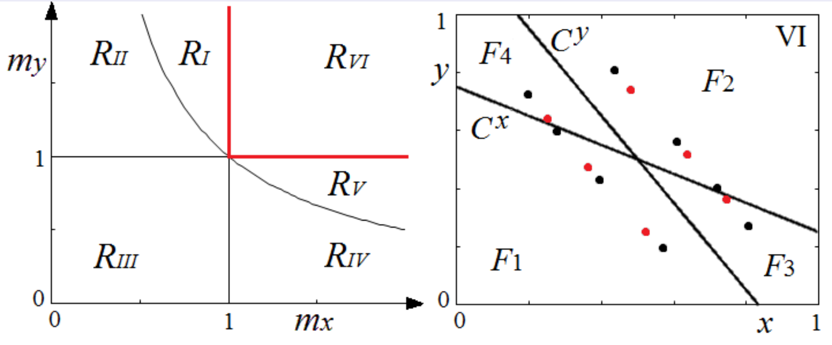

Attractors of map F (Case VI)

Several coexisting attracting interior cycles of even periods.

Border collision bifurcations of an interior cycle

Let F has an interior cycle $\Gamma_{2n} = \{p_i\}_{i=0}^{2n-1} = \{(x_i, y_i)\}_{i=0}^{2n-1}$, $n \geq 1$. It can be represented by a symbolic sequence $\sigma = \sigma_0\sigma_1...\sigma_{2n-1}$ where $\sigma_i \in \{1, 2, 3, 4\}$ and

$$\sigma_i = \begin{cases} 1 & \text{if } p_i \in D_1 \\ 2 & \text{if } p_i \in D_2 \\ 3 & \text{if } p_i \in D_3 \\ 4 & \text{if } p_i \in D_4 \end{cases}$$

- Any interior cycle Γ_{2n}, $n \geq 1$, of map F can be represented by the symbolic sequence $1^k4^m2^k3^m$ where $k \geq 1$, $0 \leq m \leq k$.

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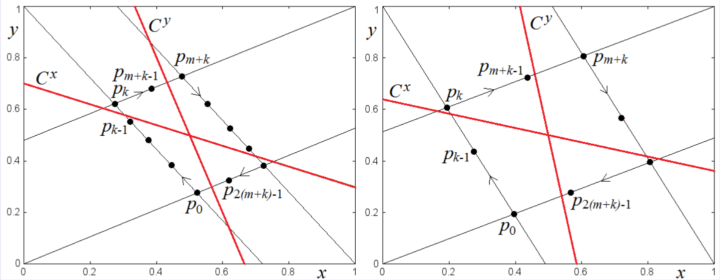

Border collision bifurcations of an interior cycle

- The rightmost point $p_0 \in D_1$ of the cycle $\Gamma_{2n} = \{p_i\}_{i=0}^{2n-1}$, $n \geq 1$, of map F with symbolic sequence $1^k 4^m 2^k 3^m$ has the following coordinates:
  $(x_0, y_0) = \left(\frac{a^m}{1+a^{m+k}}, \frac{a^{m+k}}{1+a^{m+k}} \right), \quad a := 1 - \delta$

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Proposition 1. Let $0 < \delta < 1$, $(m_x, m_y) \in R_V \cup R_{VI}$. Let F have a cycle $\Gamma_{2n} = \{p_i\}_{i=0}^{2n-1}$, $n \geq 1$. Then it is an **interior** $2(m+k)$**-cycle** having the symbolic sequence $1^k 4^m 2^k 3^m$, where $k \geq m$, $0 \leq m \leq l$, $l = \lfloor \log_{1-\delta} 0.5 \rfloor + 1$. The related **periodicity region** $P_{m,k}$ is confined by at most four BCB boundaries:

$B^1_{m,k}: \quad m_y = \frac{a^{m+k}-1}{1+a^m(a^k-2)} =: m^1_{y(m,k)} \qquad (p_0 \in C^y)$

$B^2_{m,k}: \quad m_y = \frac{a^{m+k-1}(2-a)-1}{1+a^{m-1}(a^{k+1}-2)} =: m^2_{y(m,k)} \qquad (p_{2(k+m)-1} \in C^y)$

$B^3_{m,k}: \quad m_x = \frac{1-a^{m+k}}{1+a^k(a^m-2)} =: m^3_{x(m,k)} \qquad (p_k \in C^x)$

$B^4_{m,k}: \quad m_x = \frac{1-a^{m+k-1}(2-a)}{1+a^{k-1}(a^{m+1}-2)} =: m^4_{x(m,k)} \qquad (p_{k-1} \in C^x)$

The region $P_{m,k}$ can be one-side unbounded (only the the boundaries $B^2_{m,k}$, $B^3_{m,k}$ and $B^4_{m,k}$ exist) or two-side unbounded (only $B^2_{m,k}$ and $B^3_{m,k}$ exist).

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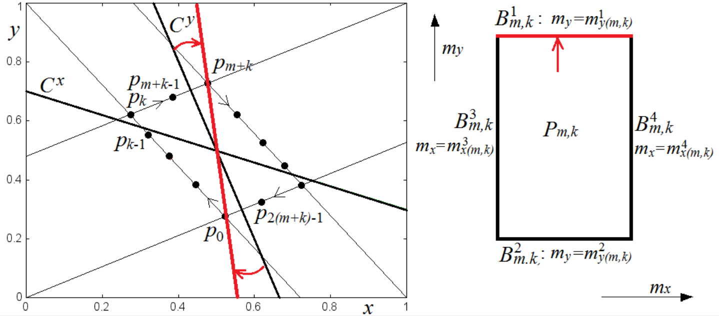
BCB of $\Gamma_{2(m+k)}$: $p_0 \in C^y$



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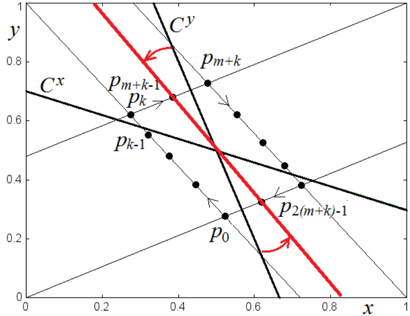

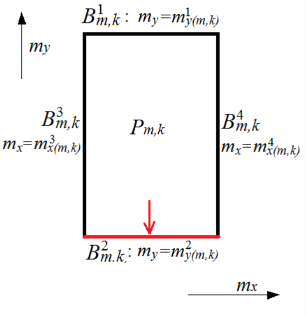

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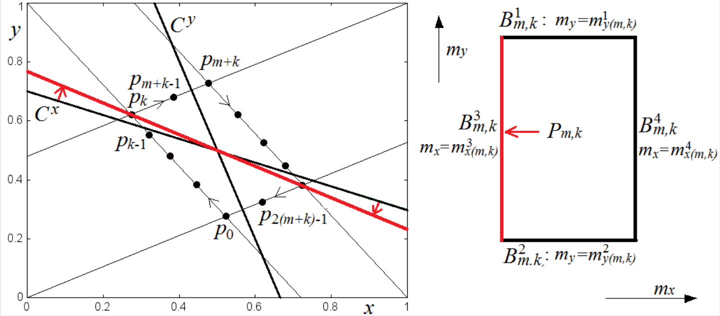

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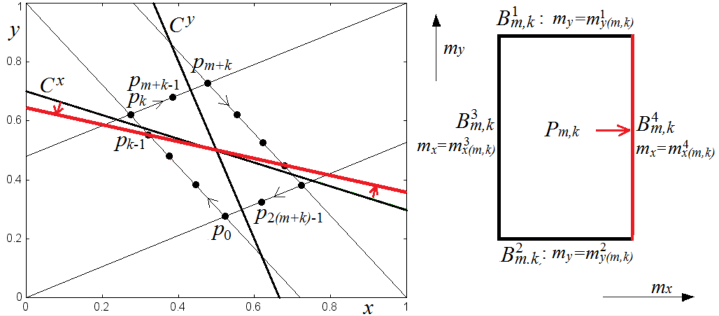

Period incrementing structures

The set of periodicity regions $\{P_{m,k}\}_{k=m}^{\infty}$ form a period incrementing structure.

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Proposition 2. The number of **period incrementing structures** in the (m_x, m_y)-plane is defined by $l = \lfloor \log_{1-\delta} 0.5 \rfloor + 1$, that is, for fixed $0 < \delta < 1$ map F can have cycles with symbolic sequences $1^k 4^m 2^k 3^m$ for any $1 \le m \le l$ and $k \ge m$.

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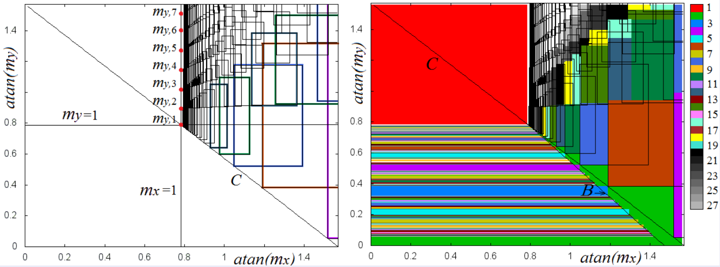
$\delta = 0.1 \Rightarrow l = 7$



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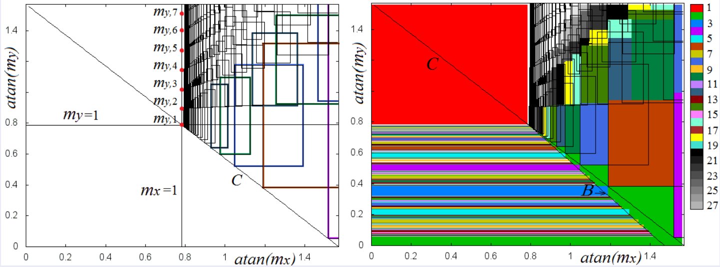
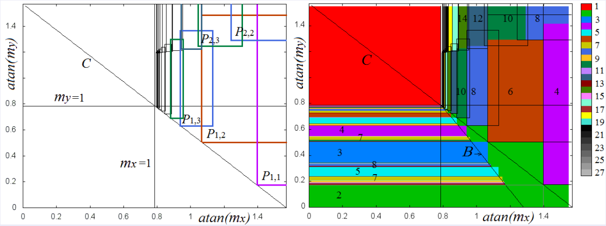


$$\lim_{k\to\infty} m^1_{y(m,k)} = \frac{1}{2(1-\delta)^m - 1} =: m_{y,m}, \quad \lim_{k\to\infty} m^2_{y(m,k)} = m_{y,m-1}$$

Period incrementing structures

$\delta = 0.3 \Rightarrow l = 2$



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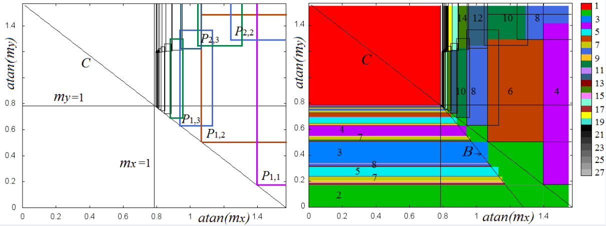
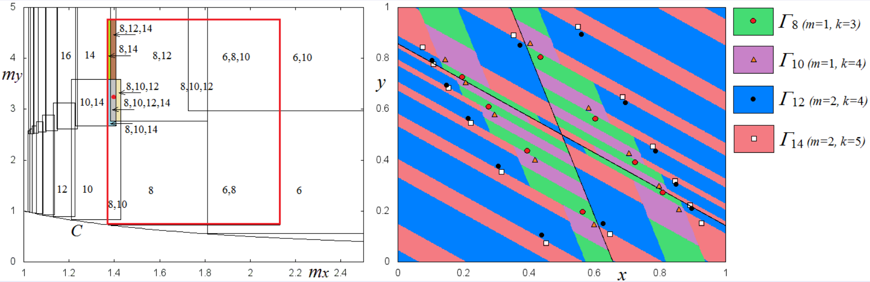


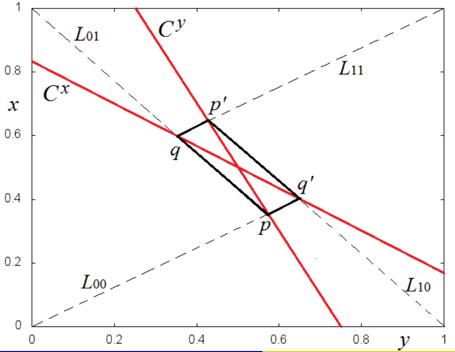


Codimension-two BCB of an interior cycle

Proposition 3. In the (x, y)-phase plane of map F a parallelogram P with vertices p, q, p' and q' can be constructed if $m_y > m_x$, $m_x > 1$, independently on δ. Here

$$p = (x_p, y_p) = (1/2 + M/2m_y, 1/2 - M/2), q = (x_q, y_q) = (1/2 - M/2, 1/2 + M/2m_x)$$

$$p' = (1 - x_p, 1 - y_p), \ q' = (1 - x_q, 1 - y_q), M = (m_y - m_x)/(m_x m_y - 1)$$



$$L_{00} : \ y = \frac{y_p}{x_p} x,$$

$$L_{01} : \ y = \frac{y_q - 1}{x_q} x + 1$$

$$L_{11} : \ y = \frac{y'_p - 1}{x'_p - 1}(x - 1) + 1,$$

$$L_{10} : \ y = \frac{y'_q}{x'_q - 1}(x - 1)$$

Codimension-two BCB of an interior cycle

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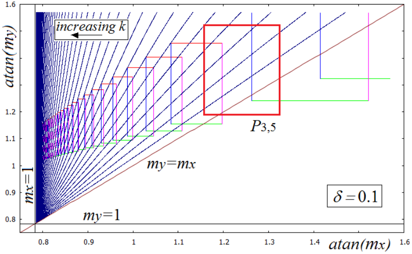
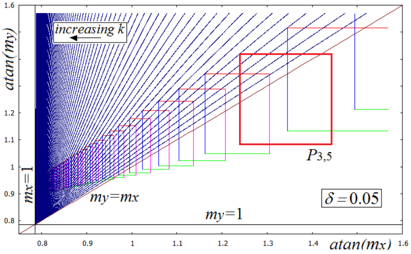
$p^{1,3}_{m,k}$ for different m, k belong to curves V_k : $m_y = \frac{a^k m_x}{m_x(a^k-1)+1}$, which for fixed k and $\delta \to 0$ ($a \to 1_-$) tend to $m_y = m_x$, while for fixed δ and $k \to \infty$ tend to $m_x = 1$.





Discrete- versus continuous-time model

Continuous-time fashion model is defined as follows:

$$\frac{d\lambda_t}{dt} \in \begin{cases} \{\alpha(1-\lambda_t)\} & \text{if } P_t > 0, \\ [-\alpha\lambda_t, \alpha(1-\lambda_t)] & \text{if } P_t = 0, \\ \{-\alpha\lambda_t\} & \text{if } P_t < 0, \end{cases}$$

$$\frac{d\lambda_t^*}{dt} \in \begin{cases} \{\alpha(1-\lambda_t^*)\}, & \text{if } P_t^* > 0, \\ [-\alpha\lambda_t^*, \alpha(1-\lambda_t^*)], & \text{if } P_t^* = 0, \\ \{-\alpha\lambda_t^*\}, & \text{if } P_t^* < 0, \end{cases}$$

$$P_t = (\lambda_t - 1/2) + m(\lambda_t^* - 1/2), \quad P_t^* = (\lambda_t^* - 1/2) + m^*(\lambda_t - 1/2).$$

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λ_t (λ_t^*) is a fraction of Conformists (Nonconformists) that chooses one of two strategies, $m > 0$ ($m^* > 0$) is the relative frequency of intergroup matching to intragroup matching from a C's (N's) point of view, $\alpha > 0$ is the speed of adjustment.

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The (m, m^*)-plane can be subdivided into the regions according to the location of the discontinuity lines. In Matsuyama, 1992, these regions are distinguished as **Case 1** ($m < 1 < mm^*$), **Case 2** ($m < mm^* < 1$), **Case 3** ($mm^* < m < 1$), **Case 4** ($m > 1 > mm^*$), **Case 5** ($m > mm^* > 1$) and **Case 6** ($mm^* > m > 1$), where **Case 6a** ($m \geq m^* > 1$) and **Case 6b** ($m^* > m > 1$).

Discrete- versus continuous-time model

Attractors of the continuous-time fashion model (Matsuyama, 1992)

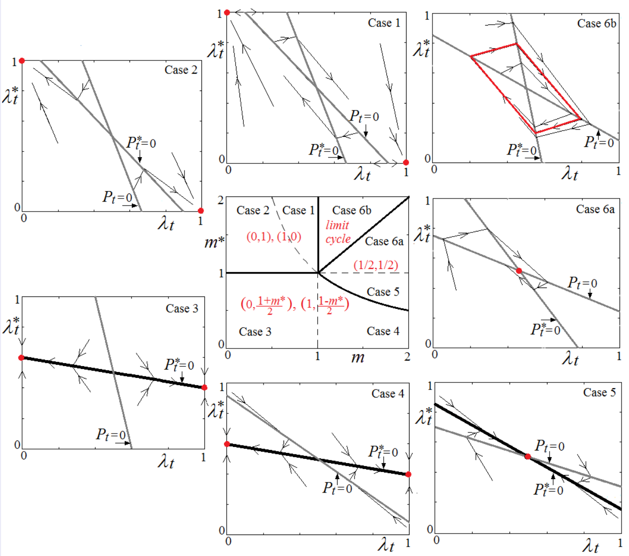
- In Case 1 and Case 2 the attractors of Λ are the **border fixed points** $(\lambda_t, \lambda_t^*) = (0, 1)$ and $(\lambda_t, \lambda_t^*) = (1.0)$.
- In Case 3 and Case 4.the attractors are the **border points** $(\lambda_t, \lambda_t^*) = \left(0, \frac{1+m^*}{2}\right)$ and $(\lambda_t, \lambda_t^*) = \left(1, \frac{1-m^*}{2}\right)$.
- In Case 5 and Case 6a the attractor is the **interior point** $(\lambda_t, \lambda_t^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$.
- In Case 6b the attractor is a **limit cycle** formed by a parallelogram with vertices

$$\begin{aligned} P &= \left(\frac{1}{2} + \frac{X_\infty}{2m^*}, \frac{1}{2} - \frac{X_\infty}{2}\right), \quad Q = \left(\frac{1}{2} - \frac{X_\infty}{2}, \frac{1}{2} + \frac{X_\infty}{2m}\right) \\ P' &= \left(\frac{1}{2} - \frac{X_\infty}{2m^*}, \frac{1}{2} + \frac{X_\infty}{2}\right), \quad Q = \left(\frac{1}{2} + \frac{X_\infty}{2}, \frac{1}{2} - \frac{X_\infty}{2m}\right) \end{aligned}$$

where

$$X_\infty = \frac{m^* - m}{mm^* - 1}$$

Discrete- versus continuous-time model



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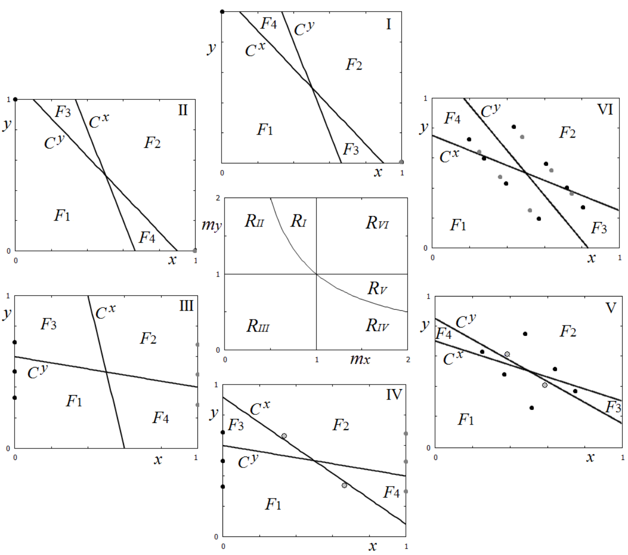

Discrete- versus continuous-time model

Cases III and IV

As $\delta \to 0$ cycles $\gamma_n \in I_0$ and $\gamma'_n \in I_1$ of map F shrink to $d = C^x \cap I_0$ and $d' = C^x \cap I_1$, respectively, while Γ_2 shrinks to S.

Discrete- versus continuous-time model

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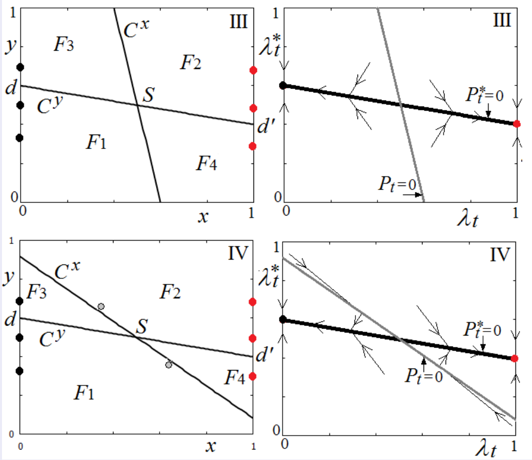

Discrete- versus continuous-time model

Cases V and VIa

As $\delta \to 0$ limit sets of trajectories of map F shrink to S.

Discrete- versus continuous-time model

Cases V and VIa

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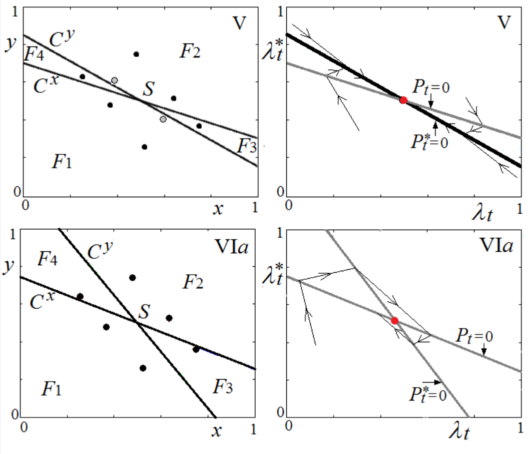

Discrete- versus continuous-time model

Cases VIb

As $\delta \to 0$ limit sets of trajectories of map F tend to parallelogram P.

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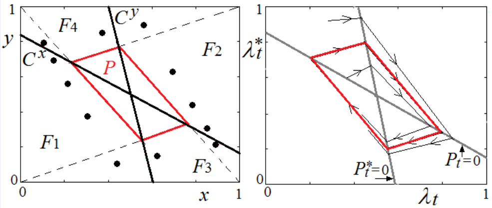

Cases VIb

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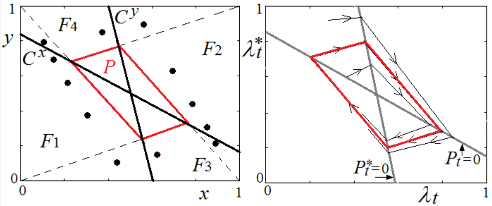
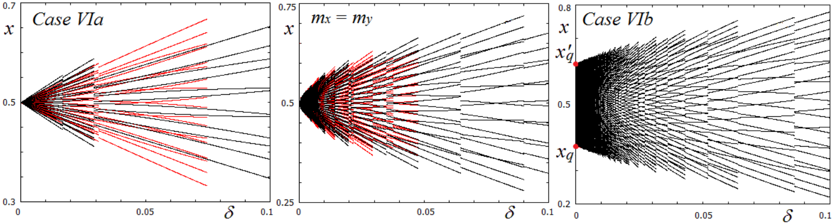


1D diagrams δ versus x for $m_y < m_x$, $m_y = m_x$ and $m_y > m_x$



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