Selection and Sorting of Heterogeneous Firms Through Competitive Pressures

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Introduction
Competitive Pressures on Heterogeneous Firms

How do competitive pressures affect selection of firms with different productivity? Or sorting across different markets?

- Melitz (2003): monopolistic competition (MC) with heterogeneous firms under CES
  - MC firms sell their products at an exogenous & common markup rate, unresponsive to competitive pressures
  - Market size: no effect on distribution of firm types and on their behaviors; All adjustments at the extensive margin.
- Melitz-Ottaviano (2008) depart from CES using Linear DS + the outside competitive sector

We depart from CES using H.S.A. (Homothetic with a Single Aggregator) DS with gross substitutes

- Homothetic (unlike the linear DS and most other commonly-used non-CES DSs)
  - a single measure of market size; the demand composition does not matter.
  - isolate the effect of endogenous markup rate from nonhomotheticity
  - straightforward to use it as a building block for multi-sector GE models
- Nonparametric and flexible (unlike CES and translog, which are special cases)
  - can be used to perform robustness-check for CES and translog
  - allow for (but no need to impose) the choke price, Marshall’s 2nd law as well as what we call the 3rd law
- Tractable due to Single Aggregator (unlike Kimball, which needs two aggregators), a sufficient statistic for competitive pressures, which acts like a magnifier of firm heterogeneity. A simple diagram for
  - proving the existence & the uniqueness of free-entry equilibrium with firm heterogeneity
  - conducting most comparative statics without parametric restrictions on the demand or productivity distribution.
    e.g., no need to assume zero overhead cost (unlike MO and ACDR)
- Defined by the market share function, for which data is readily available and easily identifiable.
Symmetric H.S.A. (Homothetic with a Single Aggregator) with Gross Substitutes
Here we consider a continuum of varieties ($\omega \in \Omega$), gross substitutes, and symmetry (Our 2017 paper for a general analysis)

Market Share of $\omega \in \Omega$ depends solely on its single relative price (= its own price/the common price aggregator)

$$\frac{p_{\omega}x_{\omega}}{px} = \frac{\partial \ln P(p)}{\partial \ln p_{\omega}} = s\left(\frac{p_{\omega}}{A(p)}\right),$$

where $$\int_\Omega s\left(\frac{p_{\omega}}{A(p)}\right) d\omega \equiv 1.$$

- $s: \mathbb{R}^+ \to \mathbb{R}^+$: the market share function, decreasing in the relative price for $s(z) > 0$ with $\lim_{z \to z^\circ} s(z) = 0$.
  - If $z^\circ \equiv \inf\{z > 0 | s(z) = 0\} < \infty$, $zA(p)$ is the choke price.

- $A(p)$: the common price aggregator defined implicitly by the adding-up constraint $$\int_\Omega s(p_{\omega}/A) d\omega \equiv 1.$$

  By construction, market shares add up to one; $A(p)$ linear homogenous in $p$ for a fixed $\Omega$. A larger $\Omega$ reduces $A(p)$.

CES if $s(z) = \gamma z^{1-\sigma}, (\sigma > 1)$; translog cost if $s(z) = -\gamma \ln \left(\frac{z}{z^\circ}\right)$; CoPaTh if $s(z) = \gamma \left[1 - \left(\frac{z}{z^\circ}\right)^{\frac{1}{\rho}}\right]^{\frac{1}{1-\rho}}, (0 < \rho < 1)$.

Unit Cost Function: $P(p) \propto A(p) \exp \left\{-\int_\Omega \left[\int_{p_{\omega}/A(p)}^z s(\xi) d\xi\right] d\omega\right\}$

Note: Our 2017 paper proved that $P(p)$ is quasi-concave and that $P(p)/A(p) \neq c$ for any $c > 0$, unless CES

✓ $A(p)$, the inverse measure of competitive pressures, fully captures cross price effects in the demand system
✓ $P(p)$, the inverse measure of TFP, captures the productivity consequences of price changes
Monopolistic Competition under H.S.A.: Pricing

Pricing (Lerner) Formula

\[ p_{\psi} \left[ 1 - \frac{1}{\zeta(p_{\psi}/A)} \right] = \psi, \]

where

\[ \zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} \equiv 1 - \mathcal{E}_s(z) > 1, \quad \text{for } z \in (0, \bar{z}); \quad \lim_{z \to \bar{z}} \zeta(z) = - \lim_{z \to \bar{z}} \mathcal{E}_s(z) = \infty, \text{if } \bar{z} < \infty. \]

\( \psi \): firm-specific marginal cost (in labor, the numeraire)

\( A = A(p) \): the inverse measure of competitive pressures, common across firms, a sufficient statistic.

Relative price

\[ z_{\psi} \equiv \frac{p_{\psi}}{A} = \frac{Z(\frac{\psi}{A})}{A}, \quad \text{an increasing function of } \psi/A, \text{ the normalized cost, only.} \]

Price elasticity

\[ \zeta \left( Z \left( \frac{\psi}{A} \right) \right) \equiv \sigma \left( \frac{\psi}{A} \right) > 1 \]

Markup rate

\[ \mu_{\psi} \equiv \frac{p_{\psi}}{\psi} = \frac{\sigma(\psi/A)}{\sigma(\psi/A) - 1} \equiv \mu \left( \frac{\psi}{A} \right) > 1 \]

Pass-through rate

\[ \rho_{\psi} \equiv \frac{\partial \ln p_{\psi}}{\partial \ln \psi} = \frac{d \ln Z(\psi/A)}{d \ln (\psi/A)} \equiv \mathcal{E}_Z \left( \frac{\psi}{A} \right) \equiv \rho \left( \frac{\psi}{A} \right) = 1 + \mathcal{E}_\mu \left( \frac{\psi}{A} \right) \]

are all functions of \( \psi/A \) only, continuously differentiable under mild regularity conditions.

More competitive pressures, a lower \( A \), act like a magnifier of firm heterogeneity.
Monopolistic Competition under H.S.A.: Revenue, Profit, & Employment

Revenue
\[ R_\psi = s(z_\psi)L = s \left( \frac{\psi}{A} \right)L \equiv r \left( \frac{\psi}{A} \right)L \] \[ \Rightarrow \mathcal{E}_r \left( \frac{\psi}{A} \right) = -\left[ \sigma \left( \frac{\psi}{A} \right) - 1 \right] \rho \left( \frac{\psi}{A} \right) < 0 \]

(Gross) Profit
\[ \Pi_\psi = \frac{s(z_\psi)L}{\zeta(z_\psi)} \equiv \frac{r(\psi/A)}{\sigma(\psi/A)}L \equiv \pi \left( \frac{\psi}{A} \right)L \] \[ \Rightarrow \mathcal{E}_\pi \left( \frac{\psi}{A} \right) = 1 - \sigma \left( \frac{\psi}{A} \right) < 0 \]

(Variable) Employment
\[ \psi x_\psi = R_\psi - \Pi_\psi = \frac{r(\psi/A)}{\mu(\psi/A)}L \equiv \ell \left( \frac{\psi}{A} \right)L \] \[ \Rightarrow \mathcal{E}_\ell \left( \frac{\psi}{A} \right) = 1 - \sigma \left( \frac{\psi}{A} \right) \rho \left( \frac{\psi}{A} \right) \leq 0 \]

- Revenue \( r(\psi/A)L \), profit \( \pi(\psi/A)L \), employment \( \ell(\psi/A)L \) all functions of \( \psi/A \), multiplied by market size \( L \), continuously differentiable under mild regularity conditions.
- Market size affects the relative profit, revenue, and employment across firms only through its effects on \( A \).
- Both revenue \( r(\psi/A)L \) and profit \( \pi(\psi/A)L \) are always strictly decreasing in \( \psi/A \).
- Employment \( \ell(\psi/A)L \) may be nonmonotonic in \( \psi/A \).
  - If the markup rate declines with \( \psi/A \), employment cannot decline as fast as the revenue.
  - If the markup rate declines faster than the revenue, the employment is increasing in \( \psi/A \).

Again, more competitive pressures, a lower \( A \), act like a magnifier of firm heterogeneity.
General Equilibrium: Existence and Uniqueness

As in Melitz, firms pay the entry cost $F_e > 0$ to draw $\psi \sim G(\psi)$; cdf with the support, $(\psi, \bar{\psi}) \subset (0, \infty)$, and pay the overhead $F > 0$ to stay & produce.

Cutoff Rule: stay if $\psi < \psi_c$; exit if $\psi > \psi_c$, where

$$\pi \left( \frac{\psi_c}{A} \right) L = F$$

positively-sloped $A \downarrow$ (more competitive pressures) $\Rightarrow$ $\psi_c \downarrow$ (tougher selection)

Free Entry Condition: 

$$F_e = \int_{\psi}^{\psi_c} \left[ \pi \left( \frac{\psi}{A} \right) L - F \right] dG(\psi)$$

negative-sloped. $A \downarrow$ (more competitive pressures) and $\psi_c \downarrow$ (tougher selection) both make entry less attractive.

$A = A(p)$ and $\psi_c$: uniquely determined, respond continuously to $F_e/L \ & F/L$ under mild regularity conditions. (This proof of unique existence applies also to the Melitz model under CES.)

With $A$ and $\psi_c$ determined, the adding-up constraint pins down masses of entrants, $M$ and of active firms, $MG(\psi_c)$. 
Cross-Sectional Implications of Marshall’s 2\textsuperscript{nd} Law

(A2): $\zeta(z_p)$ is increasing in $z_p \equiv \frac{p_p}{A} = Z(\psi/A)$

- **Price elasticity** $\zeta(Z(\psi/A)) \equiv \sigma(\psi/A)$ increasing in $\psi/A$; high-$\psi$ firms operate at more elastic parts of demand curve.
  
  - Markup Rate, $\mu(\psi/A)$, decreasing in $\psi/A \Leftrightarrow \varepsilon_\mu(\psi/A) < 0$
    high-$\psi$ firms charge lower markup rates.

  - **Incomplete Pass-Through**: The pass-through rate, $\rho(\psi/A) = 1 + \varepsilon_\mu(\psi/A) < 1$.

- **Procompetitive effect of entry/Strategic complementarity in pricing**, $\frac{\partial \ln p_p}{\partial \ln A} = 1 - \rho(\psi/A) > 0$.
  Firms set the price lower under more competitive pressures ($A = A(p) \downarrow$), due to either a larger $\Omega$ and/or a lower $p$

- **Profit**, $\pi(\psi/A)L$, always decreasing, **strictly log-supermodular** in $\psi$ and $A$.
  $A \downarrow \rightarrow$ a proportionately larger decline in profit for high-$\psi$ firms $\rightarrow$ Larger dispersion of profit

$\checkmark$ $f(\psi/A)$ is (strictly) log-super(sub)modular in $\psi & A \Leftrightarrow \varepsilon_f \left(\frac{\psi}{A}\right) \equiv \frac{d \ln f(\psi/A)}{d \ln (\psi/A)}$ is (strictly) decreasing (increasing) in $\psi/A$. 
Cross-Sectional Implications of Marshall’s 3\textsuperscript{rd} Law

In addition to A2, if we further assume, with some empirical support,

\[(A3): \rho(\psi/A) = 1 + \epsilon_{\mu}(\psi/A) \text{ is weak(strictly) increasing--we call it Weak (Strong) Marshall’s 3\textsuperscript{rd} Law.}\]

Under translog, \(\rho(\psi/A)\) is strictly decreasing, violating A3

- **Markup rate**, \(\mu(\psi/A)\), decreasing under A2, log-submodular in \(\psi \& A\) under A3. For strong A3, it is strict and \(A \downarrow \rightarrow \) a proportionately smaller decline in markup rate for high-\(\psi\) firms \(\rightarrow\) smaller dispersion of markup rate

- **Revenue**, \(r(\psi/A)L\), always decreasing, strictly log-supermodular in \(\psi \& A\) under weak A3
  \(A \downarrow \rightarrow \) a proportionately larger decline in revenue for high-\(\psi\) firms \(\rightarrow\) Larger dispersion of revenue

- **Employment**, \(\ell(\psi/A)L = \frac{r(\psi/A)}{\mu(\psi/A)}L\), hump-shaped in \(\psi/A\), strictly log-supermodular in \(\psi \& A\) under weak A3
  Employment is increasing in \(\psi\) across all active firms with a large enough overhead/market size ratio.
  \(A \downarrow \rightarrow\) Employment up for the most productive firms.

- **Pass-through rate**, \(\rho(\psi/A)\), strictly log-submodular in \(\psi \& A\) for a small enough \(\bar{Z}\) under strong A3
  \(A \downarrow \rightarrow\) a proportionately smaller increase in the pass-through rate for low-\(\psi\) firms among the active.
**Cross-Sectional Implications of More Competitive Pressures (A ↓)**

**Profit Function:** \( \Pi_\psi = \pi(\psi/A)L \)
- *always* decreasing in \( \psi \)
- strictly log-supermodular *under A2*

→ \( A \downarrow \) with \( L \) fixed shifts down with a steeper slope at each \( \psi \);
→ \( A \downarrow \) due to \( L \uparrow \), a parallel shift up, a single-crossing.

**Markup Rate Function:** \( \mu_\psi = \mu(\psi/A) > 1 \)
- decreasing in \( \psi \) *under A2*
- weakly log-submodular *under Weak A3*
- strictly log-submodular *under Strong A3*

→ \( A \downarrow \) shifts down with a flatter slope at each \( \psi \)

\[
\ln \Pi_\psi = \ln \pi(\frac{\psi}{A}) + \ln L
\]

\[
\ln \mu_\psi = \ln \mu(\frac{\psi}{A}) > 0
\]

✓ With \( \ln \psi \) in the horizontal axis, \( A \downarrow \) causes a parallel leftward shift of the graphs in these figures.
✓ \( f(\psi/A) \) is strictly log-super(sub)modular in \( \psi \) & \( A \leftrightarrow \ln f(\psi/A) \) is (strictly) concave(convex) in \( \ln(\psi/A) \).

Under Weak A3, \( R_\psi = r(\psi/A)L \), strictly log-supermodular and shares similar properties with \( \pi(\psi/A)L \).
Employment Function: $\ell(\psi/A)L = r(\psi/A)L/\mu(\psi/A)$
- Hump-shaped in $\psi$ under $A2$ and weak $A3$.
  $\rightarrow A \downarrow$ shifts up (down) for a low (high) $\psi$ with $A \downarrow$
- Strictly log-supermodular under weak $A3$
  for $A \downarrow$ with a fixed $L$; for $A \downarrow$ caused by $L \uparrow$
  Single-crossing even with a fixed $L$

Pass-Through Rate Function: $\rho_\psi = \rho(\psi/A)$
- $\rho(\psi/A) < 1$ under $A2$, hence it cannot be strictly log-submodular for a higher range of $\psi/A$
- Increasing in $\psi$ under Strong $A3$
- Strictly log-submodular for a lower range of $\psi/A$ under $A2$ and Strong $A3 \Rightarrow A \downarrow$ shifts up with a steeper slope at each $\psi$ with a small enough $\bar{z}$.

In summary, more competitive pressures ($A \downarrow$)
- $\mu(\psi/A) \downarrow$ under $A2$ & $\rho(\psi/A) \uparrow$ under strong $A3$
- Profit, Revenue, Employment become more concentrated among the most productive.
GE Comparative Statics Implications: Selection (in a single-market setting)

Effects of $F_e \downarrow$

Effects of $L \uparrow$ if $\sigma'(\cdot) > 0$ (i.e., A2)

Effects of $F \downarrow$ if $\ell'(\cdot) > 0$

- $L \uparrow$ under A2: the profit up for low-$\psi$ and down for high-$\psi$. (Similarly on the revenue under A2 and the weak A3)
- All 3 cases lead to $\psi_c \downarrow$ & $A \downarrow$, creating a non-trivial composition effect
  - Under A2, $A \downarrow$ causes $\mu(\psi/A) \downarrow$ for each $\psi$, but $\psi_c \downarrow$ means high-$\psi$ firms with lower $\mu(\psi/A)$ drop out.
  - Under strong A3, $A \downarrow$ causes $\rho(\psi/A) \uparrow$ for each $\psi$, but $\psi_c \downarrow$ means high-$\psi$ firms with higher $\rho(\psi/A)$ drop out.

The average markup (or pass-through) rate can go either way, with $F_e \downarrow$ + Pareto-productivity a knife-edge case

More competition, which causes more concentration, may result in the rise of markup.

- The effects on $M$ & $MG(\psi_c)$ depend on whether $E_G(\psi) \equiv \psi g(\psi) / G(\psi)$ is decreasing, constant, or increasing.
GE Implications: Sorting (in a multi-market setting)

More competitive pressures in larger markets:

\[ L_1 > L_2 > \cdots > L_J > 0 \implies 0 < A_1 < A_2 < \cdots < A_J < \infty \]

Under A2, more efficient firms sort themselves into larger markets with firms \( \psi \in (\psi_{j-1}, \psi_j) \) entering market-\( j \).

The composition effect: examples with Pareto-productivity such that

- The average markup rates higher (and the average pass-through rates lower under Strong A3) in larger (more competitive) markets
- A decline in \( F_e \) causes uniform declines in \( \psi_j \) & \( A_j \) with the average markup/pass-through rates unchanged.

A caution against testing A2/A3 by comparing the average markup/pass-through rates in cross-section of cities.

Markup Rate across markets

\[
\mu_j = \frac{\pi(\psi/A_j)}{\pi(\psi/A_{j-1})L_{j-1}}
\]

Pass-Through Rate across markets

\[
\rho_j = \frac{\pi(\psi/A_{j-1})}{\pi(\psi/A_j)A_j}
\]
(Highly Selective) Literature Review

**H.S.A. Demand System**: Matsuyama-Ushchev (2017)

**MC with Heterogeneous Firms**: Melitz (2003) and many others: Melitz-Redding (2015) for a survey

**MC under non-CES demand systems**: Thisse-Ushchev (2018) for a survey

- **Nonhomothetic non-CES**:
  - \( U = \int_\Omega u(x_\omega) d\omega \): Dixit-Stiglitz (77), Behrens-Murata (07), ZKPT (12), Mrázová-Neary(17), Dhingra-Morrow (19); ACDR (19)
  - **Linear-demand system**: Ottaviano-Tabuchi-Thisse (2002)


**Selection of Heterogeneous Firms through Competitive Pressures**
Melitz-Ottaviano (2008), Baqaee-Fahri-Sangani (2021)

**Sorting of Heterogeneous Firms Across Markets in General Equilibrium**

**Sorting of Heterogeneous Firms Across Markets in Reduced Form/Partial Equilibrium**

**Log-Super(Sub)modularity**: Costinot (2009), Costinot-Vogel (2010, 2015)
Selection of Heterogeneous Firms: A Single-Market Setting
A Static, Closed Economy Version of Melitz (2003), extended to H.S.A.

**Households:** supply labor (numeraire) by $L$, consume the **final good** by $X$ with the budget constraint, $PX = L$.

**Final Good Producers:** competitive, assemble **intermediate inputs** $\omega \in \Omega$, using **CRS technology**

- **Production Function:**
  \[ X = X(x) \equiv \min_p \left\{ px = \int_\Omega p_\omega x_\omega d\omega \mid P(p) \geq 1 \right\} \]

- **Unit Cost Function:**
  \[ P = P(p) \equiv \min_x \left\{ px = \int_\Omega p_\omega x_\omega d\omega \mid X(x) \geq 1 \right\} \]

**Note:** Either $X(x)$ or $P(p)$ can be a primitive of CRS technology, as long as linear homogeneity, monotonicity and quasi-concavity are satisfied.

- **Demand Curve for** $\omega$: $x_\omega = X(x) \frac{\partial p(p)}{\partial p_\omega}$; **Inverse Demand Curve for** $\omega$: $p_\omega = P(p) \frac{\partial X(x)}{\partial x_\omega}$

- **Market Size:** $px = P(p)X(x) = L$

**Note:** This is due to the one-market setting. In a multi-market extension later, size of each market differs from $L$. 
Symmetric H.S.A. (Homothetic with a Single Aggregator) with Gross Substitutes

**Market Share** of $\omega$ depends *solely* on its single relative price (= its own price/the *common* price aggregator)

$$\frac{p_\omega x_\omega}{p x} = \frac{\partial \ln P(p)}{\partial \ln p_\omega} = s \left( \frac{p_\omega}{A(p)} \right),$$

where $$\int_{\Omega} s \left( \frac{p_\omega}{A(p)} \right) d\omega \equiv 1.$$

- $s: \mathbb{R}_{++} \to \mathbb{R}_{+}$: the *market share function*, decreasing in the *relative price*; $C^3$ for $s(z) > 0$ with $\lim_{z \to z^\circ} s(z) = 0$.
  - If $z^\circ \equiv \inf\{z > 0 | s(z) = 0\} < \infty$, $z^\circ A(p)$ is the *choke price*.

- $A(p)$: the *common price aggregator* defined implicitly by the *adding up constraint* $\int_{\Omega} s(p_\omega/A) d\omega \equiv 1$.
  - By construction, market shares add up to one; $A(p)$ linear homogenous in $p$ for a fixed $\Omega$. A larger $\Omega$ reduces $A(p)$.

  CES if $s(z) = \gamma z^{1-\sigma}, (\sigma > 1)$; translog cost if $s(z) = -\gamma \ln \left( \frac{z}{z^\circ} \right)$; CoPaTh if $s(z) = \gamma \left[ 1 - \left( \frac{z}{z^\circ} \right)^{1-\rho} \right]^{1-\rho}, (0 < \rho < 1)$.

**Unit Cost Function:** $P(p) \propto A(p) \exp \left\{ - \int_{\Omega} \left[ \int_{p_\omega / A(p)}^{\hat{z}} s(\xi) \frac{d\xi}{\xi} \right] d\omega \right\}$

*Note:* Our 2017 paper proved that $P(p)$ is *quasi-concave* and that $P(p) / A(p) \neq c$ for any $c > 0$, unless CES

✓ $A(p)$, the inverse measure of *competitive pressures*, fully captures *cross price effects* in the demand system

✓ $P(p)$, the inverse measure of TFP, captures the *productivity consequences* of price changes
Monopolistically Competitive Intermediate Inputs Producers $\omega \in \Omega$

**Timing:** the same with Melitz.

- Sunk cost of entry, $F_e > 0$. (All costs are paid in labor.)
- Each entrant draws its marginal cost $\psi \sim G(\cdot) \in C^3$, with the density $G'(\psi) = g(\psi) > 0$ on $(\psi, \bar{\psi}) \subseteq (0, \infty)$. 
  
  $E_g(\psi) \equiv \psi g(\psi) / G(\psi) \in C^2$ and $E_g(\psi) \equiv \psi g'(\psi) / g(\psi) \in C^1$.

MC firms are ex-post heterogeneous *only* in $\psi$.

Firm productivity, $1/\psi = \varphi \sim 1 - G(1/\varphi)$ with the density $g(1/\varphi) / \varphi^2 > 0$ on $(\varphi, \bar{\varphi}) \subseteq (0, \infty)$.

- Each firm decides either to exit without producing or to stay & produce with an overhead cost, $F > 0$.
- Firms that stay will sell their products at the profit-maximizing prices.

**Pricing Behaviors of MC firms** after drawing $\psi_\omega$:

Each firm takes $A = A(p)$ and $px = L$ given.

$$
\max_{p_\omega}(p_\omega - \psi_\omega)x_\omega = \max_{\psi_\omega < p_\omega \leq z \bar{A}} \left(1 - \frac{p_\omega}{A}\right) s\left(\frac{p_\omega}{A}\right)L = \max_{\psi_\omega / A < z_\omega \leq \bar{z}} \left(1 - \frac{\psi_\omega / A}{z_\omega}\right)s(z_\omega)L
$$

where $z_\omega \equiv p_\omega / A$ is the relative price.
FOC: 
\[ z_\omega \left[ 1 - \frac{1}{\zeta(z_\omega)} \right] = \frac{\psi_\omega}{A} \]

Price Elasticity Function 
\[ \zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} \equiv 1 - \varepsilon_s(z) > 1, \]
for \( z \in (0, \bar{z}) \) with \( \lim_{z \to \bar{z}} \zeta(z) = - \lim_{z \to \bar{z}} \varepsilon_s(z) = \infty \), if \( \bar{z} \) is finite. The markup rate is \( \zeta(z_\omega)/(\zeta(z_\omega) - 1) \).

We maintain the following *regularity* assumption for ease of exposition.

| (A1) | For all \( z \in (0, \bar{z}) \),  
| | \( \varepsilon_{z(\zeta-1)/\zeta(z)} > 0 \iff \varepsilon_{z/\zeta}\left(\zeta-1\right)(z) < 1 \iff \varepsilon_{z/\zeta}(z) = \varepsilon_s(z) - \varepsilon_\zeta(z) < 0 \) |

- (A1) means that \( \zeta(z)/(\zeta(z) - 1) \) cannot go up as fast as \( z \).  
  \( \rightarrow \) (A1) holds with decreasing \( \zeta(\cdot)/(\zeta(\cdot) - 1) \leftrightarrow \) increasing \( \zeta(\cdot) \), i.e., under Marshall’s 2\textsuperscript{nd} Law, introduced later.
- (A1) means the marginal revenue is strictly increasing in \( p_\omega \) (hence strictly decreasing in \( x_\omega \)).  
  \( \rightarrow \) FOC determines the profit maximizing \( z_\omega \) as an increasing \( C^2 \) function of \( \psi_\omega / A \).
  \( \rightarrow \) Firms with the same \( \psi \) set the same price, earn the same profit \( \rightarrow \) we index firms by \( \psi \), as \( p_\psi, z_\psi \equiv p_\psi / A \).
- (A1) ensures that the maximized profit \( s(\cdot)L/\zeta(\cdot) \) is a decreasing \( C^2 \) function of \( \psi_\omega / A \).

Without (A1), the maximizing price would be piecewise-continuous (i.e., the price would jump up at some values of \( \psi \)) and the maximized profit would be piecewise-differentiable, which would complicate the analysis.
Monopoly Pricing: Markup and Pass-Through Rates

Lerner Formula:

\[ z_{\psi} \left[ 1 - \frac{1}{\zeta(z_{\psi})} \right] = \frac{\psi}{A} \]

Under A1, the Inverse Function Theorem allows us to express it as

Relative Price:

\[ z_{\psi} = Z \left( \frac{\psi}{A} \right) \in (\psi/A, \bar{z}), \]

from which

Price Elasticity:

\[ \zeta(z_{\psi}) = \zeta \left( Z \left( \frac{\psi}{A} \right) \right) \equiv \sigma \left( \frac{\psi}{A} \right) > 1 \]

Markup Rate:

\[ \mu_{\psi} \equiv \frac{p_{\psi}}{\psi} = \frac{\zeta(z_{\psi})}{\zeta(z_{\psi}) - 1} = \frac{\sigma(\psi/A)}{\sigma(\psi/A) - 1} \equiv \mu \left( \frac{\psi}{A} \right) \in (1, \bar{z}A/\psi) \]

Pass-Through Rate:

\[ \rho_{\psi} \equiv \frac{\partial \ln p_{\psi}}{\partial \ln \psi} = \frac{\partial \ln Z(\psi/A)}{\partial \ln(\psi/A)} \equiv \xi Z \left( \frac{\psi}{A} \right) = \frac{1}{1 - \xi \zeta/(\zeta - 1)(Z(\psi/A))} \equiv \rho \left( \frac{\psi}{A} \right) = 1 + \xi \mu \left( \frac{\psi}{A} \right) > 0 \]

Notes:

- Relative price, and markup rate, all \( C^2 \) functions of \( \psi/A \) only.
  - Relative price, always strictly increasing in \( \psi/A \)
  - Markup rate, strictly decreasing in \( \psi/A \) under A2
- Pass-through rate, a \( C^1 \) function of \( \psi/A \); only, strictly increasing in \( \psi/A \) under strong A3.
Revenue, Profit, and Employment

Revenue

\[ R_\psi \equiv p_\psi x_\psi = s(z_\psi) L = s \left( Z \left( \frac{\psi}{A} \right) \right) L \equiv r \left( \frac{\psi}{A} \right) L \quad \Rightarrow \quad \varepsilon_r \left( \frac{\psi}{A} \right) = \left[ 1 - \sigma \left( \frac{\psi}{A} \right) \right] \rho \left( \frac{\psi}{A} \right) < 0 \]

(Gross) Profit

\[ \Pi_\psi \equiv \left( 1 - \frac{\psi}{A} \right) s(z_\psi) L = \frac{s(z_\psi)}{\zeta(z_\psi)} L = \frac{r(\psi/A)}{\sigma(\psi/A)} L \equiv \pi \left( \frac{\psi}{A} \right) L \quad \Rightarrow \quad \varepsilon_\pi \left( \frac{\psi}{A} \right) = 1 - \sigma \left( \frac{\psi}{A} \right) < 0 \]

(Variable) Employment

\[ \psi x_\psi = R_\psi - \Pi_\psi = \left[ 1 - \frac{1}{\zeta(z_\psi)} \right] s(z_\psi) L = \left[ r \left( \frac{\psi}{A} \right) - \pi \left( \frac{\psi}{A} \right) \right] L = \frac{r(\psi/A)}{\mu(\psi/A)} L \equiv \ell \left( \frac{\psi}{A} \right) L \]

\[ \Rightarrow \varepsilon_\ell \left( \frac{\psi}{A} \right) = \varepsilon_r \left( \frac{\psi}{A} \right) - \varepsilon_\mu \left( \frac{\psi}{A} \right) = 1 - \sigma \left( \frac{\psi}{A} \right) \rho \left( \frac{\psi}{A} \right) \leq 0 \]

- Revenue, profit, employment all \( C^2 \) functions of \( \psi/A \), multiplied by market size \( L \).
- Market size affects the relative profit, revenue, and employment across firms only through its effects on \( A \).
- Elasticities of revenue, profit, employment w.r.t. \( \psi/A \), all \( C^1 \) functions of \( \psi/A \) only.

- Both revenue \( r(\psi/A) L \) and profit \( \pi(\psi/A) L \) are always strictly decreasing in \( \psi/A \).
- Employment \( \ell(\psi/A) L \) may be nonmonotonic in \( \psi/A \).
  - If the markup rate declines with \( \psi/A \), employment cannot decline as fast as the revenue.
  - If the markup rate declines faster than the revenue, the employment is increasing in \( \psi/A \).
**Equilibrium Conditions:** Assume $F + F_e < \pi(0)L$.

**Cutoff Rule:** Stay if $\psi < \psi_c$; exit if $\psi > \psi_c$, where

$$\pi \left( \frac{\psi_c}{A} \right) L = F \iff \frac{\psi_c}{A} = \pi^{-1} \left( \frac{F}{L} \right)$$

depicted by a positively-sloped ray.

$A \downarrow$ (more competitive pressures) $\implies \psi_c \downarrow$ (tougher selection)

**Free Entry Condition:**

$$F_e = \int_{\psi}^{\psi_c} \left[ \pi \left( \frac{\psi}{A} \right) L - F \right] dG(\psi)$$

negative-sloped, both $A \downarrow$ (more competitive pressures) and $\psi_c \downarrow$ (tougher selection) make entry less attractive.

$A$ and $\psi_c$: uniquely determined as $C^2$ functions of $F_e/L$ & $F/L$ with the interior solution, $0 < G(\psi_c) < 1$ for

$$0 < \frac{F_e}{L} < \int_{\psi}^{\psi_c} \left[ \pi \left( \pi^{-1} \left( \frac{F}{L} \frac{\psi}{A} \right) - \frac{F}{L} \right) \right] dG(\psi),$$

which holds for a sufficiently small $F_e > 0$ with no further restrictions on $G(\cdot)$ and $s(\cdot)$.

(This unique existence proof does not assume A2 and hence applies also to the Melitz model under CES.)
From the adding-up constraint, \( 1 \equiv \int_{\Omega} s(p_\omega/A) d\omega = M \int_{\psi}^{\psi_c} s(z_\psi) dG(\psi) = M \int_{\psi}^{\psi_c} r(\psi/A) dG(\psi) \),

Mass of entrants

\[
M = \left[ \int_{\psi}^{\psi_c} r \left( \frac{\psi}{A} \right) dG(\psi) \right]^{-1} = \left[ \int_{\xi}^{1} r \left( \pi^{-1} \left( \frac{F}{L} \right) \xi \right) dG(\psi_c \xi) \right]^{-1} > 0
\]

Mass of active firms

\[
MG(\psi_c) = \left[ \int_{\psi}^{\psi_c} r \left( \frac{\psi}{A} \right) dG(\psi) \right]^{-1} = \left[ \int_{\xi}^{1} r \left( \pi^{-1} \left( \frac{F}{L} \right) \xi \right) d\tilde{G}(\xi; \psi_c) \right]^{-1} > 0
\]

where \( \xi \equiv \psi/\psi_c \) and \( \tilde{G}(\xi; \psi_c) \equiv G(\psi_c \xi) / G(\psi_c) \) is the cdf of \( \xi \), conditional on \( \xi \leq 1 \).

**Lemma 1:** \( \mathcal{E}_g'(\psi) < 0 \Rightarrow \mathcal{E}_G'(\psi) < 0 \) generally; \( \mathcal{E}_g'(\psi) \geq 0 \Rightarrow \mathcal{E}_G'(\psi) \geq 0 \), with some additional conditions.

**Lemma 2:** A lower \( \psi_c \) (tougher selection) shifts \( \tilde{G}(\xi; \psi_c) \) to the right (left)

- in the MLR ordering if \( \mathcal{E}_g'(\psi) < (>)0 \).
- in the FSD ordering if \( \mathcal{E}_G'(\psi) < (>)0 \).

\( \tilde{G}(\xi; \psi_c) \) is independent of \( \psi_c \) if \( \mathcal{E}_g(\psi) \) and \( \mathcal{E}_G(\psi) \) are constant, i.e., \( G(\cdot) \) is a power \( \Leftrightarrow F(\cdot) \) is a Pareto

A lower \( \psi_c \) (tougher selection) shifts \( \tilde{G}(\xi; \psi_c) \) to the right if Fréchet, Weibull, or Lognormal.

**Lemma 4:** The integrals in the equilibrium conditions are finite and hence the equilibrium is well-defined, if

\[
\psi > 0 \Leftrightarrow \overline{\varphi} < \infty \quad \text{or} \quad 1 + \lim_{z \to 0} \zeta(z) < 2 + \lim_{\psi \to 0} \mathcal{E}_g(\psi) = - \lim_{\varphi \to \infty} \mathcal{E}_f(\varphi) < \infty \quad \text{for} \quad \underline{\varphi} = 0 \Leftrightarrow \overline{\varphi} = \infty.
\]
CES Benchmark: Revisiting Melitz
**CES Benchmark:** For all \( z \in (0, \infty), \zeta(z) = \sigma > 1 \Leftrightarrow s(z) = \gamma z^{1-\sigma}. \)

**Pricing:** 
\[
p_{\psi} \left(1 - \frac{1}{\sigma}\right) = \psi \Leftrightarrow \mu \left(\frac{\psi}{A}\right) = \frac{\sigma}{\sigma - 1} > 1 \Rightarrow \rho \left(\frac{\psi}{A}\right) = 1
\]
Markup rate constant; Pass-through rate equal to one.

**Cutoff Rule:**
\[c_0 L \left(\frac{\psi_c}{A}\right)^{1-\sigma} = F,\]

**Free Entry Condition:**
\[
\int_{\psi}^{\psi_c} \left[c_0 L \left(\frac{\psi}{A}\right)^{1-\sigma} - F\right] dG(\psi) = F_e
\]
with \( c_0 > 0. \) As \( L \) changes, the intersection moves along
\[
\int_{\psi}^{\psi_c} \left[\left(\frac{\psi}{\psi_c}\right)^{1-\sigma} - 1\right] dG(\psi) = \frac{F_e}{F}
\]
\( F_e / F \downarrow \) and a FSD shift of \( G(\cdot) \) to the left \( \Rightarrow \psi_c \downarrow \) (tougher selection). \( \psi_c \) unaffected by \( L \), and independent of \( A.\)

\[
A = \psi_c \left(\frac{c_0 L}{F}\right)^{1-\sigma} = \left(\frac{c_0 L}{F_e} \int_{\psi}^{\psi_c} \left[(\psi)^{1-\sigma} - (\psi_c)^{1-\sigma}\right] dG(\psi)\right)^{\frac{1}{1-\sigma}}.
\]

\( L \uparrow, F_e \downarrow, F \downarrow, \) a FSD shift of \( G(\cdot) \) to the left \( \Rightarrow A \downarrow \) (more competitive pressures)
CES Benchmark (Continue)

Revenue of a ψ-firm:
\[ r \left( \frac{\psi}{A} \right) L = \sigma c_0 L \left( \frac{\psi}{A} \right)^{1-\sigma} = \sigma F \left( \frac{\psi}{\psi_c} \right)^{1-\sigma} \geq \sigma F \]

(Gross) Profit of a ψ-firm:
\[ \pi \left( \frac{\psi}{A} \right) L = c_0 L \left( \frac{\psi}{A} \right)^{1-\sigma} = F \left( \frac{\psi}{\psi_c} \right)^{1-\sigma} \geq F \]

(Variable) Employment of a ψ-firm:
\[ \ell \left( \frac{\psi}{A} \right) L = (\sigma - 1) c_0 L \left( \frac{\psi}{A} \right)^{1-\sigma} = (\sigma - 1) F \left( \frac{\psi}{\psi_c} \right)^{1-\sigma} \geq (\sigma - 1) F \]

All decreasing power functions of ψ with
\[ \mathcal{E}_r \left( \frac{\psi}{A} \right) = \mathcal{E}_\pi \left( \frac{\psi}{A} \right) = \mathcal{E}_\ell \left( \frac{\psi}{A} \right) = 1 - \sigma < 0. \]

Relative size of two firms with ψ, ψ′ ∈ (ψ, ψ_c), whether measured in the profit, employment, and revenue, unaffected by L, F_e, F, G(·), as well as A and ψ_c, and thus never change across equilibriums.
CES Benchmark (Continue)

Mass of entrants

\[ M = \frac{L/\sigma}{F_e + G(\psi_c)F} = \frac{L}{\sigma F_e} \left[ 1 - \frac{1}{H(\psi_c)} \right] \]

Mass of active firms

\[ MG(\psi_c) = \frac{L/\sigma}{F_e/G(\psi_c) + F} = \frac{L}{H(\psi_c)\sigma F} \]

where \( H(\psi_c) \equiv \int_{\xi}^{1}(\xi)^{1-\sigma}G(\xi; \psi_c) \). Since \((\xi)^{1-\sigma}\) is decreasing, \( H'(\psi_c) > (<)0 \) if \( \varepsilon_G'(\psi) < (>)0 \) (Lemma 2).

Hence,

**Proposition 1:** Under CES,

- \( L \uparrow \) keeps \( \psi_c \) unaffected; increases both \( M \) and \( MG(\psi_c) \) proportionately;
- \( F_e \downarrow \) reduces \( \psi_c \); increases \( M \); increases (decreases) \( MG(\psi_c) \) if \( \varepsilon_G'(\psi) < (>)0 \);
- \( F \downarrow \) increases \( \psi_c \); increases \( MG(\psi_c) \); increases (decreases) \( M \) if \( \varepsilon_G'(\psi) < (>)0 \);

A FSD shift of \( G(\cdot) \) to the left reduces \( \psi_c \) with ambiguous effects on \( M \) and \( MG(\psi_c) \), even if \( G(\cdot) \) is a power.

**Effects of Market Size \( L \) under CES:**

- No effect on the markup rate.
- No effect on the cutoff, \( \psi_c \)
- No effect on the distribution of productivity, revenue, and employment across firms.
- Masses of entrants and of active firms change proportionately. All adjustments at the extensive margin.
Marshall’s 2\textsuperscript{nd} and 3\textsuperscript{rd} Laws: Cross-Sectional Implications
**Marshall’s 2nd Law (A2)**

(A2): \( \zeta'(z) > 0 \) for all \( z \in (0, \bar{z}) \) \( \iff \) \( \sigma'(\psi/A) > 0 \) for all \( \psi/A \in (0, \bar{z}) \)

Note: \( A2 \Rightarrow A1 \).

**Lemma 5:** For a positive-valued function of a single variable, \( \psi/A > 0 \),

\[
\text{sgn} \left\{ \frac{\partial^2 \ln f(\psi/A)}{\partial \psi \partial A} \right\} = -\text{sgn} \left\{ \mathcal{E}_f \left( \frac{\psi}{A} \right) \right\} = -\text{sgn} \left\{ \frac{d^2 \ln f \left( e^{\ln(\psi/A)} \right)}{(d \ln(\psi/A))^2} \right\}
\]

\( f(\psi/A) \) log-super(sub)modular in \( \psi \) & \( A \) \( \iff \) \( \mathcal{E}_f (\cdot) < (>) 0 \) \( \iff \) \( \ln f \left( e^{\ln(\psi/A)} \right) \) concave (convex) in \( \ln(\psi/A) \)

**Proposition 2:** Under \( A2 \),

**Incomplete Pass-Through**

\[
0 < \rho \left( \frac{\psi}{A} \right) = 1 + \mathcal{E}_\mu \left( \frac{\psi}{A} \right) = 1 - \mathcal{E}_{1/\mu} \left( \frac{\psi}{A} \right) < 1
\]

Less efficient firms operate at more elastic parts of demand and have lower markup rates

**Procompetitive Effect**/

**Strategic Complementarity in Pricing**

\[
\frac{\partial \ln p_\psi}{\partial \ln A} = 1 - \rho \left( \frac{\psi}{A} \right) = -\mathcal{E}_\mu \left( \frac{\psi}{A} \right) = \mathcal{E}_{1/\mu} \left( \frac{\psi}{A} \right) > 0
\]

More competitive pressures (\( A \downarrow \) due to entry or lower prices of competing products) \( \Rightarrow \) lower prices/markup rates.

**Strict Log-Supermodular Profit:**

\[
\mathcal{E}_\pi' \left( \frac{\psi}{A} \right) < 0 \iff \frac{\partial^2 \ln \pi(\psi/A)}{\partial \psi \partial A} > 0
\]

More competitive pressures (\( A \downarrow \)) \( \Rightarrow \) a proportionately larger decline in the profit among high-\( \psi \) firms \( \Rightarrow \) a larger dispersion of the profit across firms; more concentration of profits among the productive.
**Marshall’s 3rd Law (A3):**

(A3) **Weak (Strong) Marshall’s 3rd Law of demand.** For all $z \in (0, \bar{z})$,

\[
\varepsilon'_{\zeta/(\zeta-1)}(z) = -\frac{d}{dz}\left(\frac{z\zeta'(z)}{[\zeta(z) - 1]\zeta(z)}\right) \geq (>0) \iff \rho'\left(\frac{\psi}{A}\right) = \varepsilon'_{\zeta/\zeta}(\psi) = \varepsilon'_{\mu}(\psi) \geq (>0)
\]

Strong A3 $\Rightarrow$ The markup rate declines at the lower rate for higher $z$ $\Rightarrow$ The pass-through rate higher for higher $\psi$.


**Proposition 3:** Under A3(A3),

- **Weak (Strict) Log-Submodular Markup Rate:**
  \[
  \varepsilon'_{Z}(\frac{\psi}{A}) = \rho'\left(\frac{\psi}{A}\right) = \varepsilon'_{\mu}(\psi) \geq (>0) \iff \frac{\partial^2 \ln(Z(\psi/A))}{\partial \psi \partial A} = \frac{\partial^2 \ln(\mu(\psi/A))}{\partial \psi \partial A} \leq (<0),
  \]

  For the strict case, more competitive pressures ($A \downarrow$) $\Rightarrow$ proportionately smaller rate decline among high-$\psi$ firms. $\Rightarrow$ a smaller dispersion of the markup rate across firms.

  Under A2+A3

- **Strict Log-Supermodular Revenue:**
  \[
  \varepsilon'_{r}(\frac{\psi}{A}) < 0 \iff \frac{\partial^2 \ln r(\psi/A)}{\partial \psi \partial A} > 0
  \]

  **Strict Log-Supermodular Employment:**
  \[
  \varepsilon'_{e}(\frac{\psi}{A}) = \varepsilon'_{r}(\frac{\psi}{A}) - \varepsilon'_{\mu}(\psi) < 0 \iff \frac{\partial^2 \ln e(\psi/A)}{\partial \psi \partial A} > 0.
  \]

  More competitive pressures ($A \downarrow$) $\Rightarrow$ proportionately larger decline in the revenue among high-$\psi$ firms $\Rightarrow$ a larger dispersion of the revenue across firms; more concentration of revenue among the productive.
A2+A3: Cross-Sectional Implications of $A \downarrow$ on Profit and Markup Rate

**Profit Function:** $\Pi_{\psi} = \pi(\psi/A)L$
- always decreasing in $\psi$
- strictly log-supermodular under A2
→ $A \downarrow$ with $L$ fixed shifts down with a steeper slope at each $\psi$;
→ $A \downarrow$ due to $L \uparrow$, a parallel shift up, a single-crossing

**Markup Rate Function:** $\mu_{\psi} = \mu(\psi/A) > 1$
- decreasing in $\psi$ under A2
- weakly log-submodular under Weak A3
- strictly log-submodular under Strong A3
→ $A \downarrow$ shifts down with a flatter slope at each $\psi$

$\ln \Pi_{\psi} = \ln \pi \left( \frac{\psi}{A} \right) + \ln L$

$\ln \mu_{\psi} = \ln \mu \left( \frac{\psi}{A} \right) > 0$

- With $\ln \psi$ in the horizontal axis, $A \downarrow$ causes a parallel leftward shift of the graphs in these figures.
- $f(\psi/A)$ is strictly log-super(sub)modular in $\psi$ and $A$ iff $\ln f(e^x)$ is concave(convex) in $x$.

Under Weak A3, $R_{\psi} = r(\psi/A)L$, is strictly log-supermodular and shares similar properties with $\pi(\psi/A)L$. 
Lemma 6: Under A2 and the weak A3, $\lim_{\psi/A \to 0} \rho(\psi/A)\sigma(\psi/A) < 1 < \lim_{\psi/A \to \bar{z}} \rho(\psi/A)\sigma(\psi/A)$.

Since A2+A3 also implies $E'_\ell(\psi/A) < 0$,

Proposition 4: Under A2 and the weak A3, the employment function, $\ell(\psi/A) = \frac{r(\psi/A)}{\mu(\psi/A)}$ is hump-shaped, with its unique peak is reached at, $\hat{z} \equiv Z(\hat{\psi}/A) < \bar{z}$, where

$$E_{s(\zeta-1)/\zeta}(\hat{z}) = 0 \iff \frac{\hat{z}'(\hat{z})}{\zeta(\hat{z})} = [\zeta'(\hat{z}) - 1]^2 \iff E_\ell\left(\frac{\hat{\psi}}{A}\right) = 0 \iff \rho\left(\frac{\hat{\psi}}{A}\right)\sigma\left(\frac{\hat{\psi}}{A}\right) = 1.$$

A2+A3 are sufficient but not necessary for being hump-shaped.

Corollary of Proposition 4: Employments across active firms are

- increasing in $\psi$ if $\psi_c < \hat{\psi} \iff F/L > \pi\left(\hat{\psi}/A\right) = \pi(Z^{-1}(\hat{z}))$;
  This occurs when the overhead/market size ratio is sufficiently high.
- hump-shaped in $\psi$ if $\psi < \hat{\psi} < \psi_c \iff F/L < \pi\left(\hat{\psi}/A\right) = \pi(Z^{-1}(\hat{z}))$ & $A > \frac{\psi}{Z^{-1}(\hat{z})}$.

Employments are decreasing among the most productive firms.

- decreasing in $\psi$, if $\hat{\psi} < \psi \iff A < \frac{\psi}{Z^{-1}(\hat{z})}$, which is possible only if $\psi > 0$.

Proposition 5: Suppose that A2 and the strong A3 hold, so that $0 < \rho(\psi/A) < 1$ and $\rho(\psi/A)$ is strictly increasing. Then, $\rho(\psi/A)$ is strictly log-submodular for all $\psi/A < \bar{z}$ with a sufficiently small $\bar{z}$.
**Employment Function:** $\ell(\psi / A) L = r(\psi / A) L / \mu(\psi / A)$

- **Hump-shaped in** $\psi$ **under** $A2$ **and weak** $A3$.
  - $A \downarrow$ shifts up (down) for a low (high) $\psi$ with $A \downarrow$
- **Strictly log-supermodular** **under Weak** $A3$
  for $A \downarrow$ with a fixed $L$; for $A \downarrow$ caused by $L \uparrow$
**Single-crossing even with a fixed** $L$

**Pass-Through Rate Function:** $\rho_{\psi} = \rho(\psi / A)$

- $\rho(\psi / A) < 1$ **under** $A2$, **hence it cannot be strictly** log-submodular for a higher range of $\psi / A$
- **Strictly increasing in** $\psi$ **under Strong** $A3$
- **Strictly log-submodular for a lower range of** $\psi / A$ **under** $A2$ **and Strong** $A3 \Rightarrow A \downarrow$ shifts up with a steeper slope at each $\psi$ **with a small enough** $Z$.

In summary, more competitive pressures ($A \downarrow$)

- $\mu(\psi / A) \downarrow$ **under** $A2$ & $\rho(\psi / A) \uparrow$ **under strong** $A3$
- Profit, Revenue, Employment become more concentrated among the most productive.
Comparative Statics: General Equilibrium Effects
Comparative Statics: Effects of $F_e$, $L$, and $F$ on $\psi_c / \psi$ and $A$

**Proposition 6:**

\[
\left[ \mathbb{E}_\sigma \left( \psi, \psi_c \right) - 1 \right] \frac{dA}{A} = \left( 1 - f_x \right) \left( \frac{dF_e}{F_e} \right) - \frac{dL}{L} + f_x \left( \frac{dF}{F} \right);
\]

\[
\left[ \mathbb{E}_\sigma \left( \psi, \psi_c \right) - 1 \right] \left( \frac{d\psi_c}{\psi_c} - \frac{dA}{A} \right) = \delta \left( \frac{dL}{L} - \frac{dF}{F} \right);
\]

\[
\left[ \mathbb{E}_\sigma \left( \psi, \psi_c \right) - 1 \right] \frac{d\psi_c}{\psi_c} = \left( 1 - f_x \right) \left( \frac{dF_e}{F_e} \right) - (1 - \delta) \left( \frac{dL}{L} \right) + (f_x - \delta) \left( \frac{dF}{F} \right)
\]

where

\[
\mathbb{E}_\sigma \left( \psi, \psi_c \right) \equiv \frac{\int_{\psi}^{\psi_c} \sigma(\psi / A) \pi(\psi / A) dG(\psi) / G(\psi_c)}{\int_{\psi}^{\psi_c} \pi(\psi / A) dG(\psi) / G(\psi_c)} = 1 + \frac{\int_{\psi}^{\psi_c} \ell(\psi / A) dG(\psi) / G(\psi_c)}{\int_{\psi}^{\psi_c} \pi(\psi / A) dG(\psi) / G(\psi_c)} > 1
\]

The profit-weighted average of $\sigma(\psi / A)$ among the active firms.

\[
f_x \equiv \frac{FG(\psi_c)}{F_e + FG(\psi_c)} = \frac{FG(\psi_c)}{L \int_{\psi}^{\psi_c} \pi(\psi / A) dG(\psi)} = \frac{\pi(\psi_c / A)}{\int_{\psi}^{\psi_c} \pi(\psi / A) dG(\psi) / G(\psi_c)} < 1
\]

The share of the overhead in the total fixed cost = the profit at the cut-off relative to the average profit among the active firms.

\[
\delta \equiv \frac{\mathbb{E}_\sigma \left( \psi, \psi_c \right) - 1}{\sigma(\psi_c / A) - 1} = \frac{\pi(\psi_c / A)}{\ell(\psi_c / A)} \int_{\psi}^{\psi_c} \ell(\psi / A) dG(\psi) / G(\psi_c) = f_x \frac{\int_{\psi}^{\psi_c} \ell(\psi / A) dG(\psi) / G(\psi_c)}{\ell(\psi_c / A)} > 0
\]

The profit/employment ratio at the cut-off to the average profit/employment ratio among the active firms.
Corollary of Proposition 6

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$\psi_c/A$</th>
<th>$\psi_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_e$</td>
<td>$\frac{dA}{dF_e} &gt; 0$</td>
<td>$\frac{d(\psi_c/A)}{dF_e} = 0$</td>
<td>$\frac{d\psi_c}{dF_e} &gt; 0$</td>
</tr>
<tr>
<td>$L$</td>
<td>$\frac{dA}{dL} &lt; 0$</td>
<td>$\frac{d(\psi_c/A)}{dL} &gt; 0$</td>
<td>$\frac{d\psi_c}{dL} &lt; 0 \Leftrightarrow \sigma(\psi_c/A) &gt; \mathbb{E}_\sigma(\psi, \psi_c)$; holds globally if $\sigma'(\cdot) &gt; 0$, i.e., A2</td>
</tr>
<tr>
<td>$F$</td>
<td>$\frac{dA}{dF} &gt; 0$</td>
<td>$\frac{d(\psi_c/A)}{dF} &lt; 0$</td>
<td>$\frac{d\psi_c}{dF} &gt; 0 \Leftrightarrow \ell(\psi_c/A) &gt; \int_{\psi}^{\psi_c} \ell(\psi/A) dG(\psi)/G(\psi_c)$; holds globally if $\ell'(\cdot) &gt; 0$</td>
</tr>
</tbody>
</table>

Effects of $F_e \downarrow$  

Effects of $L \uparrow$ if $\sigma'(\cdot) > 0$ (i.e., A2)  

Effects of $F \downarrow$ if $\ell'(\cdot) > 0$
Proposition 7: Market Size Effect on Profit, $\Pi_\psi \equiv \pi(\psi/A) L$ and Revenue, $R_\psi \equiv r(\psi/A) L$

**7a:** Under $A2$, there exists a unique $\psi_0 \in (\underline{\psi}, \underline{\psi}_c)$ such that $\sigma \left( \frac{\psi_0}{A} \right) = \mathbb{E}_\sigma (\psi, \underline{\psi}_c)$ with

$$
\frac{d \ln \Pi_\psi}{d \ln L} > 0 \iff \sigma \left( \frac{\psi}{A} \right) < \mathbb{E}_\sigma (\psi, \underline{\psi}_c) \text{ for } \psi \in (\underline{\psi}, \psi_0),
$$

and

$$
\frac{d \ln \Pi_\psi}{d \ln L} < 0 \iff \sigma \left( \frac{\psi}{A} \right) > \mathbb{E}_\sigma (\psi, \underline{\psi}_c) \text{ for } \psi \in (\psi_0, \underline{\psi}_c).
$$

**7b:** Under $A2$ and the weak $A3$, there exists $\psi_1 > \psi_0$, such that

$$
\frac{d \ln R_\psi}{d \ln L} > 0 \text{ for } \psi \in (\underline{\psi}, \psi_1).
$$

Furthermore, $\psi_1 \in (\psi_0, \underline{\psi}_c)$ and

$$
\frac{d \ln R_\psi}{d \ln L} < 0 \text{ for } \psi \in (\psi_1, \underline{\psi}_c),
$$

for a sufficiently small $F$. 
By putting together the main implications of Propositions 2, 3, 6, and 7

$F_e \downarrow$ under A2 and the weak A3

$A \downarrow$, $\psi_c \downarrow$ with $\psi_c / A$ unchanged

The cutoff firms before the change and the cutoff firms after the change have

- the same markup rate $\mu(\psi_c / A)$
- the same profit $\pi(\psi_c / A)L = F$
- the same revenue, $r(\psi_c / A)L$
$L \uparrow$ under A2 and the weak A3

$A \downarrow$, $\psi_c \downarrow$ with $\psi_c/A \uparrow$ and $\sigma(\psi_c/A) \uparrow$

Compared to the cutoff firms before the change, the cutoff firms after the change have
- a lower markup rate, $\mu(\psi_c/A) \downarrow$
- the same profit, $\pi(\psi_c/A)L = F$.
- a higher revenue, $r(\psi_c/A)L = \sigma(\psi_c/A)F \uparrow$

Profits up (down) for firms with $\psi < (>\psi_0$; Revenues up (down) for firms with $\psi < (>\psi_1$ for a sufficiently small $F$. 

\[
\ln \mu(\psi_c/A) = \ln \mu \left(\frac{\psi}{A}\right)
\]

\[
\ln \psi_c
\]

\[
\ln \pi(\psi_c/A)L = \ln \pi \left(\frac{\psi}{A}\right)L
\]

\[
\ln \sigma(\psi_c/A)F = \ln \sigma \left(\frac{\psi}{A}\right)F
\]

\[
\ln \mu(\psi_c/A)
\]

\[
\ln \psi_c
\]

\[
\ln \psi_c
\]

\[
\ln \psi_c
\]

\[
\ln \psi_c
\]
$F \downarrow$ under A2 and the weak A3 with $\ell'(\cdot) > 0$

$A \downarrow$, $\psi_c \downarrow$ with $\psi_c/A \uparrow$ and $\sigma(\psi_c/A) \uparrow$

Compared to the cutoff firms before the change, the cutoff firms after the change have
- a lower markup rate, $\mu(\psi_c/A) \downarrow$
- a lower profit, $\pi(\psi_c/A)L = F \downarrow$.
- a lower revenue, $r(\psi_c/A)L = \sigma(\psi_c/A)F \downarrow$. 
Average Markup and Pass-Through Rates in a Single Market Model: The Composition Effect

- Under A2, $A \downarrow$ causes $\mu(\psi/A) \downarrow$ for each $\psi$, but $\psi_c \downarrow$ means high-$\psi$ firms with lower $\mu(\psi/A)$ drop out.
- Under strong A3, $A \downarrow$ causes $\rho(\psi/A) \uparrow$ for each $\psi$, but $\psi_c \downarrow$ means high-$\psi$ firms with higher $\rho(\psi/A)$ drop out.

Both the average markup and pass-through rates can go either way.

**Proposition 8a.** Assume $\underline{\psi} = 0$. Consider a shock, which causes a proportional decline in $A$ and $\psi_c$, so that $\psi_c/A$ remains constant. Then, for any weighting function $w(\psi/A)$,

i) the weighted average of any monotonically decreasing (increasing) $f(\psi/A)$ decreases (increases) if $\mathcal{E}_g'(\cdot) < 0$ and increases (decreases) if $\mathcal{E}_g'(\cdot) > 0$.

ii) the weighted average of any $f(\psi/A)$, monotonic or not, remains constant, if $\mathcal{E}_g'(\cdot) = 0$.

From part ii) of Proposition 8a: $F_e \downarrow$ under $G(\psi) = (\psi/\underline{\psi})^k$ offers a knife-edge case, where the average markup and pass-through rates, whether weighted by the revenue, the profit, or the employment, remain unchanged.

**Proposition 8b.** Assume that A2 holds, $\underline{\psi} = 0$, and $\ell(\psi/A)$ is increasing in $\psi/A$ for all $\psi/A \in (0, \psi_c/A)$. Consider a shock, which causes a proportional decline in $A$ and $\psi_c$, so that $\psi_c/A$ remains constant. Then, the $\ell(\cdot)$-weighted average markup rate decreases if $\mathcal{E}_g'(\cdot) < 0$; remains constant if $\mathcal{E}_g'(\cdot) = 0$; and increases if $\mathcal{E}_g'(\cdot) > 0$.

*More competition, which causes more concentration, may result in the rise of markup.*
Proposition 9: Comparative Statics on $\mathit{MG}(\psi_c)$, $\mathit{M/L}$, $\mathit{MG}(\psi_c)/L$

<table>
<thead>
<tr>
<th>Proposition 9a (The Effects of $F_e$ on $M$ and $\mathit{MG}(\psi_c)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dM}{dF_e} &lt; 0$; \hspace{1cm} $\mathcal{E}_G'(\psi) \geq 0, \forall \psi \in (\psi, \overline{\psi}) \Rightarrow \frac{d[\mathit{MG}(\psi_c)]}{dF_e} \leq 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Proposition 9b (The Effects of $L$ on $M$ and $\mathit{MG}(\psi_c)$): Under A2,</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dM}{dL} &gt; 0$; \hspace{1cm} $\mathcal{E}_G'(\psi) \leq 0, \forall \psi \in (\psi, \overline{\psi}) \Rightarrow \frac{d[\mathit{MG}(\psi_c)]}{dL} &gt; 0$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Proposition 9c (The Effects of $L$ on $\mathit{M/L}$ and $\mathit{MG}(\psi_c)/L$): Under A2,</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(\psi) = (\psi/\overline{\psi})^\kappa \Rightarrow \frac{d}{dL}\left(\frac{M}{L}\right) &gt; 0$ \hspace{1cm} $\mathcal{E}_G'(\psi) \geq 0, \forall \psi \in (\psi, \overline{\psi}) \Rightarrow \frac{d[\mathit{MG}(\psi_c)]}{dL} &lt; 0$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Proposition 9d (The Effects of $F$ on $M$ and $\mathit{MG}(\psi_c)$): If $\ell'(\cdot) &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dM}{dF} &lt; 0$; \hspace{1cm} $\mathcal{E}_G'(\psi) \leq 0, \forall \psi \in (\psi, \overline{\psi}) \Rightarrow \frac{d[\mathit{MG}(\psi_c)]}{dF} &lt; 0$.</td>
</tr>
</tbody>
</table>
Limit Case of $F \to 0$ with $\bar{z} < \infty$

**Cutoff Rule:**

$$\pi\left(\frac{\psi_c}{A}\right) = 0 \iff \frac{\psi_c}{A} = \bar{z} = \pi^{-1}(0)$$

**Free Entry Condition:**

$$\frac{F_e}{L} = \int_{\psi_c}^\psi \pi\left(\frac{\bar{z} \psi}{\psi_c}\right) dG(\psi) = \int_{\psi_c}^{\bar{z}} A \pi\left(\frac{\psi}{A}\right) dG(\psi).$$

$A$ and $\psi_c$: uniquely determined as $C^2$ functions of $F_e/L$ with the interior solution, $0 < G(\psi_c) < 1$ for

$$0 < \frac{F_e}{L} < \int_{\psi_c}^{\psi} \pi\left(\frac{\bar{z} \psi}{\psi_c}\right) dG(\psi).$$

$$\frac{d\psi_c}{\psi_c} = \frac{dA}{A} = \frac{1}{E_\sigma(\psi, \psi_c) - 1} \frac{d(F_e/L)}{F_e/L} = \frac{\int_{\psi_c}^{\psi} \pi(\psi/A) dG(\psi) / G(\psi_c) d(F_e/L)}{\int_{\psi_c}^{\psi} \ell(\psi/A) dG(\psi) / G(\psi_c) F_e/L}$$

$$\frac{dM}{d(F_e/L)} < 0;$$

$$E'_G(\psi) \leq 0 \iff \frac{d[MG(\psi_c)]}{d(F_e/L)} \leq 0$$
$F_e/L \downarrow$ for $F \to 0$ with $\bar{z} < \infty$ under A2 and the weak A3

$A \downarrow, \psi_c \downarrow$ with $\psi_c/A = \bar{z}$ unchanged.

The cutoff firms always (i.e., both before and after the change) have

- $\mu(\psi_c/A) = 1$
- $\pi(\psi_c/A) L = 0$.
- $r(\psi_c/A) L = 0$.

Profits up (down) for firms with $\psi < (>) \psi_0$;
Revenues up (down) for firms with $\psi < (>) \psi_1$.

In the middle and the lower panels,
Blue: the effects of $F_e/L \downarrow$ due to $F_e \downarrow$
Purple: the effects of $F_e/L \downarrow$ due to $L \uparrow$
Sorting of Heterogeneous Firms in a Multi-Market Setting
A Multi-Market Extension: $J$ markets, $j = 1, 2, \ldots, J$, with market size $L_j$.

Possible Interpretations
- Identical Households, whose preferences are Cobb-Douglas, $\sum_{j=1}^{J} \beta_j \ln X_j$ with $\sum_{j=1}^{J} \beta_j = 1$. Then, $L_j = \beta_j L$.
- $J$ types of consumers, with $L_j$ equal to the total income of type-$j$ consumers. “Types” can be their “tastes” or “locations”, etc.

Let’s keep it simple by assuming
- Market size is the only exogenous source of heterogeneity across markets: Index them as $L_1 > L_2 > \ldots, > L_J > 0$.
- Labor is fully mobile, equalizing the wage across the markets. We continue to use it as the *numeraire*.
- Firm’s marginal cost, $\psi$, is independent of the market it chooses.
  - Each firm pays $F_e > 0$ to draw its marginal cost $\psi \sim G(\psi)$.
  - Knowing its $\psi$, each firm decides which market to enter and produce with an overhead cost, $F > 0$, or exit without producing.
  - Firms sell their products at the profit-maximizing prices in the market they enter.

Equilibrium Condition:

$$F_e = \int_{\psi} \max\{\Pi_\psi - F, 0\} dG(\psi) = \int_{\psi} \max\left\{ \max_{1 \leq j \leq J} \{\Pi_{j\psi}\} - F, 0\right\} dG(\psi)$$

where

$$\Pi_{j\psi} \equiv \frac{s\left(Z(\psi/A_j)\right)}{\zeta\left(Z(\psi/A_j)\right)} L_j \equiv \frac{r(\psi/A_j)}{\sigma(\psi/A_j)} L_j = \pi\left(\frac{\psi}{A_j}\right) L_j$$
Proposition 10: Equilibrium Characterization under A2

**Larger markets are more competitive:**

\[ 0 < A_1 < A_2 < \cdots < A_J < \infty, \text{ where } M \int_{\psi_{j-1}}^{\psi_j} r \left( \frac{\psi}{A_j} \right) dG(\psi) = 1. \]

Note: Because \( \pi(\cdot) \) is strictly decreasing, this implies \( \pi(\psi/A_1) < \pi(\psi/A_2) < \cdots < \pi(\psi/A_J) \) for all \( \psi \).

**More productive firms self-select into larger markets (Positive Assortative Matching)**

Firms with \( \psi \in (\psi_{j-1}, \psi_j) \) enter market-\( j \) and those with \( \psi \in (\psi_j, \infty) \) do not enter any market, where

\[
0 \leq \underline{\psi} = \psi_0 < \psi_1 < \psi_2 < \cdots < \psi_J < \overline{\psi} = \infty \quad \text{ where } \quad \frac{\pi(\psi_j/A_j) L_j}{\pi(\psi_j/A_{j+1}) L_{j+1}} = 1 \quad \text{for } 1 \leq j \leq J - 1; \quad \pi(\psi_j/A_j) L_j \equiv F
\]

Note: \( \psi_j \)-firms are indifferent btw entering Market-\( j \) & entering Market-(\( j + 1 \)).

**Free Entry Condition:**

\[
\sum_{j=1}^{J} \int_{\psi_{j-1}}^{\psi_j} \left\{ \pi \left( \frac{\psi}{A_j} \right) L_j - F \right\} dG(\psi) = F_e
\]

**Mass of Firms in Market-\( j \):**

\[
M[G(\psi_j) - G(\psi_{j-1})] > 0
\]
Logic Behind Sorting

\[ L_j > L_{j+1} \implies A_j < A_{j+1}. \] Otherwise, no firm would enter \( j + 1 \).

\[ \frac{\pi(\psi/A_j)}{\pi(\psi/A_{j+1})} \text{ strictly decreasing in } \psi \]

due to strict log-supermodularity of \( \pi(\psi/A) \) under A2

\[ \implies \left[ \frac{\Pi_{j\psi}}{\Pi_{(j+1)\psi}} = \frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}} \geq 1 \iff \psi \leq \psi_j \right] \]

Under CES, \( \frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}} \) is independent of \( \psi \).

\[ \implies \frac{\Pi_{j\psi}}{\Pi_{(j+1)\psi}} = \frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}} = 1 \text{ in equilibrium.} \]

\[ \implies \text{Firms indifferent across all markets.} \]

\[ \implies \text{Distribution of firms across markets is indeterminate.} \]

Our mechanism generates sorting through competitive pressures. As such,

- complementary to agglomeration-economies-based mechanisms offered by Gaubert (2018) and Davis-Dingel (2019)
- justifies the equilibrium selection criterion used by Baldwin-Okubo (2006), which use CES, as a limit argument.
Cross-Sectional, Cross-Market Implications:

Profits: Under A2

\[ L_j > L_{j+1} \Rightarrow A_j < A_{j+1} \Rightarrow \left[ \frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}} \leq 1 \iff \psi \leq \psi_j \right] \]

\[ \Pi_\psi = \max_j \left\{ \pi \left( \frac{\psi}{A_j} \right) L_j \right\}, \text{ the upper-envelope of } \pi \left( \frac{\psi}{A_j} \right) L_j, \text{ is continuous and decreasing in } \psi, \text{ with the kinks at } \psi_j. \]

Continuous, since the lower markup rate in Market-\( j \) cancels out its larger market size, keeping \( \psi_j \)-firms indifferent btw Market-\( j \) & Market-(\( j + 1 \)).

Revenues: Under A2

\[ \frac{r(\psi_j/A_j)L_j}{r(\psi_j/A_{j+1})L_{j+1}} = \frac{\sigma(\psi_j/A_j)\pi(\psi_j/A_j)L_j}{\sigma(\psi_j/A_{j+1})\pi(\psi_j/A_{j+1})L_{j+1}} = \frac{\sigma(\psi_j/A_j)}{\sigma(\psi_j/A_{j+1})} > 1 \]

\( R_\psi \): continuously decreasing in \( \psi \) within each market; jumps down at \( \psi_j \).

With the markup rate lower in Market-\( j \), \( \psi_j \)-firms need to earn higher revenue to keep them indifferent btw Market-\( j \) & and Market-(\( j + 1 \)).
**Markup Rates: Under A2**

\[ L_j > L_{j+1} \Rightarrow A_j < A_{j+1} \Rightarrow \sigma \left( \frac{\psi_j}{A_j} \right) > \sigma \left( \frac{\psi_j}{A_{j+1}} \right) \iff \mu \left( \frac{\psi_j}{A_j} \right) < \mu \left( \frac{\psi_j}{A_{j+1}} \right) \]

\( \mu_\psi \): continuously decreasing in \( \psi \) within each market but jumps up at \( \psi_j \).

- The average markup rates may be *higher* in larger (and hence more competitive) markets.
- The average markup rates in all markets may *go up*, even if all markets become more competitive (\( A_j \downarrow \)).

**Pass-Through Rates: Under A2 and the strong A3**

\[ L_j > L_{j+1} \Rightarrow A_j < A_{j+1} \Rightarrow \rho \left( \frac{\psi_j}{A_j} \right) > \rho \left( \frac{\psi_j}{A_{j+1}} \right) \]

\( \rho_\psi \): continuously increasing in \( \psi \) within each market but jumps down at \( \psi_j \).

- The average pass-through rates may be *lower* in larger (and hence more competitive) markets.
- The average pass-through rates in all markets *go down* even if all markets become more competitive (\( A_j \downarrow \)).
Average Markup and Pass-Through Rates in a Multi-Market Model: The Composition Effect

**Proposition 11a:** Suppose A2 and $G(\psi) = (\psi/\overline{\psi})^K$. There exists a sequence, $L_1 > L_2 > \cdots > L_J > 0$, such that, in equilibrium, the weighted average of $f(\psi/A_j)$ across firms operating at market-$j$ are increasing (decreasing) in $j$ even though $f(\cdot)$ is increasing (decreasing) and hence $f(\psi/A_j)$ is decreasing (increasing) in $j$.

**Corollary of Proposition 11a:** An example with $G(\psi) = (\psi/\overline{\psi})^K$, such that the average markup rates are higher (and the average pass-through rates are lower under Strong A3) in larger markets.

**Proposition 11b:** Suppose A2 and $G(\psi) = (\psi/\overline{\psi})^K$. Then, a change in $F_e$ keeps

i) the ratios $a_j \equiv \psi_{j-1}/\psi_j$ and $b_j \equiv \psi_j/A_j$

and

ii) the weighted average of $f(\psi/A_j)$ across firms operating at market-$j$, for any weighting function $w(\psi/A_j)$, unchanged for all $j = 1, 2, \ldots, J$.

**Corollary of Proposition 11b:** $F_e \downarrow$ and $G(\psi) = (\psi/\overline{\psi})^K$ offers a knife-edge case, where the average markup and pass-through rates of all markets remain unchanged.

A caution against testing A2/A3 by comparing the average markup & pass-through rates across space and time.
Concluding Remarks (Unfinished)
Appendices
Three Classes of Homothetic Demand Systems: Matsuyama-Ushchev (2017)

- **H.S.A.** (Homotheticity with a Single Aggregator)
- **HDIA** (Homotheticity with Direct Implicit Additivity)
- **HIIA** (Homotheticity with Indirect Implicit Additivity)

We use H.S.A. because, under HDIA and HIIA, the equilibrium may not exist, or if it exists, there may be multiple equilibriums, unless we impose some strong restrictions on both productivity distributions and the price elasticity functions.
Symmetric H.S.A. with Gross Substitutes: An Alternative (Equivalent) Definition

Market Share of $\omega$ depends *solely* on its single relative quantity (= its own quantity/the *common* quantity aggregator)

$$\frac{p_\omega x_\omega}{px} = \frac{\partial \ln X(x)}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(x)}\right),$$

where

$$\int_\Omega s^*\left(\frac{x_\omega}{A^*(x)}\right)d\omega \equiv 1.$$

- $s^*: \mathbb{R}_{++} \rightarrow \mathbb{R}_+$: the *market share function*, with $0 < ys^*(y)/y < 1$, where $y$ is the *relative quantity*
  - If $\bar{z} \equiv s^*(0) = \lim_{y \rightarrow 0}[s^*(y)/y] < \infty$, $\bar{z}A(p)$ is the *choke price*.

- $A^*(x)$: the *common quantity aggregator* defined implicitly by the *adding up constraint*

$$\int_\Omega s^*(x_\omega/A^*)d\omega \equiv 1.$$

By construction, market shares add up to one; $A^*(x)$ linear homogenous in $x$ for a fixed $\Omega$. A larger $\Omega$ raises $A^*(x)$.

Two definitions equivalent with the one-to-one mapping, $s(z) \leftrightarrow s^*(y)$, defined by $s^* \equiv s(s^*/y)$ or $s \equiv s^*(s/z)$.

- CES if $s^*(y) = \gamma^{1/\sigma}y^{1-1/\sigma}$; CoPaTh if $s^*(y) = \left[(y)\frac{\rho-1}{\rho} + (yz)\frac{\rho-1}{\rho}\right]^{\rho-1}$ with $\rho \in (0,1)$.

**Production Function:** $X(x) \propto A^*(x) \exp\left\{\int_\Omega \left[\int_0^{x_\omega/A^*(x)} s^*(\xi)\frac{d\xi}{\xi}\right]d\omega\right\}$

*Note:* Our 2020 paper proved

$$\left[1 - \frac{d \ln s(z)}{d \ln z}\right]\left[1 - \frac{d \ln s^*(y)}{d \ln y}\right] = 1$$

Our 2017 paper proved that $X(x)$ is quasi-concave & that $A^*(x)/X(x) = P(p)/A(p) \neq c$ for any $c > 0$ unless CES

- $A^*(x)$, the measure of competitive pressures, fully captures cross quantity effects in the inverse demand system
- $X(x)$, the measure of output, captures the output implications of input changes
TFP

\[
\ln \left( \frac{X}{L} \right) = \ln \left( \frac{1}{P} \right) = \ln \left( \frac{1}{cA} \right) + M \int_{\psi}^{\psi_c} \left[ \int_{Z(\psi/A)}^{z} \frac{s(\xi)}{\xi} d\xi \right] dG(\psi)
\]

\[
= \ln \left( \frac{1}{cA} \right) + \frac{\int_{\psi}^{\psi_c} \left[ \int_{Z(\psi/A)}^{z} \frac{s(\xi)}{\xi} d\xi \right] dG(\psi)}{\int_{\psi}^{\psi_c} s \left( \frac{Z(\psi)}{A} \right) dG(\psi)} = \ln \left( \frac{1}{cA} \right) + \frac{\int_{\psi}^{\psi_c} \left[ \int_{Z(\psi/A)}^{z} \frac{s(\xi)}{\xi} d\xi \right] dG(\psi)}{\int_{\psi}^{\psi_c} \left[ \int_{Z(\psi/A)}^{z} \left( \frac{\xi}{A} \right) - 1 \right] \frac{s(\xi)}{\xi} d\xi \right] dG(\psi)}
\]

with a positive constant \( c > 0 \).

**Labor Market Equilibrium:** satisfied automatically from the Walras Law.

\[
\text{Labor Demand} = M \left[ F_e + \int_{\psi}^{\psi_c} \left( x_{\psi} \psi + F \right) dG(\psi) \right] = M \left[ F_e + FG(\psi_c) + \int_{\psi}^{\psi_c} \ell \left( \frac{\psi}{A} \right) LdG(\psi) \right]
\]

\[
= LM \left[ \int_{\psi}^{\psi_c} \left[ \pi \left( \frac{\psi}{A} \right) + \ell \left( \frac{\psi}{A} \right) \right] dG(\psi) \right] \quad \text{(from the Free Entry Condition)}
\]

\[
= LM \int_{\psi}^{\psi_c} r \left( \frac{\psi}{A} \right) dG(\psi) = L \quad \text{(from the Adding Up Constraint)}
\]
Three Parametric Families of H.S.A.

Generalized Translog:

\[ s(z) = \gamma \left( 1 - \frac{\sigma - 1}{\eta} \ln \left( \frac{z}{\beta} \right) \right)^{\eta} = \gamma \left( -\frac{\sigma - 1}{\eta} \ln \left( \frac{z}{\bar{z}} \right) \right)^{\eta} \quad ; \quad z < \bar{z} \equiv \beta e^{\frac{\eta}{\sigma - 1}} \]

\[ \Rightarrow \zeta(z) = 1 + \frac{\sigma - 1}{1 - \frac{\sigma - 1}{\eta} \ln \left( \frac{z}{\beta} \right)} = 1 - \frac{\eta}{\ln \left( \frac{z}{\bar{z}} \right)} > 1 \]

\[ \Rightarrow \eta z \zeta'(z) = [\zeta(z) - 1]^2 \Rightarrow \frac{z \zeta'(z)}{[\zeta(z) - 1] \zeta(z)} = \frac{1}{\eta} \left[ 1 - \frac{1}{\zeta(z)} \right] = \frac{1}{\eta - \ln \left( \frac{z}{\bar{z}} \right)} \]

satisfying A2 but violating A3.

- CES is the limit case, as \( \eta \to \infty \), while holding \( \beta > 0 \) and \( \sigma > 1 \) fixed, so that \( \bar{z} \equiv \beta e^{\frac{\eta}{\sigma - 1}} \to \infty \).
- Translog is the special case where \( \eta = 1 \).
- \( z = Z \left( \frac{\psi}{A} \right) \) is given as the inverse of \( \frac{\eta z}{\eta - \ln(z/\bar{z})} = \frac{\psi}{A} \);
- If \( \eta \geq 1 \), employment is globally decreasing in \( z \);
- If \( \eta < 1 \), employment is hump-shaped with the peak, given by \( \hat{z}/\bar{z} = \frac{\hat{\psi}}{(1 - \eta)ZA} = \exp \left[ -\frac{\eta^2}{1 - \eta} \right] < 1 \), decreasing in \( \eta \).
**Constant Pass-Through (CoPaTh):** Matsuyama-Ushchev (2020b). For $0 < \rho < 1$, $\sigma > 1$, $\bar{z} \equiv \beta \left( \frac{\sigma}{\sigma - 1} \right)^{\frac{\rho}{1-\rho}}$

$$s(z) = \gamma \sigma^{\frac{\rho}{1-\rho}} \left[ 1 - \left( \frac{z}{\bar{z}} \right)^{\frac{1-\rho}{\rho}} \right]^{\frac{\rho}{1-\rho}} \Rightarrow 1 - \frac{1}{\zeta(z)} = \left( \frac{z}{\bar{z}} \right)^{\frac{1-\rho}{\rho}} < 1 \Rightarrow \mathcal{E}_{1-1/\zeta}(z) = -\mathcal{E}_{\zeta/(\zeta-1)}(z) = \frac{1-\rho}{\rho} > 0$$

satisfying $A2$ and the weak form of $A3$ (but not the strong form). Then, for $\psi/A < \bar{z}$,

$$p_\psi = (\bar{z}A)^{1-\rho}(\psi)^\rho; \quad Z \left( \frac{\psi}{A} \right) = (\bar{z})^{1-\rho} \left( \frac{\psi}{A} \right)^\rho; \quad \sigma \left( \frac{\psi}{A} \right) = \frac{1}{1 - (\psi/\bar{z}A)^{1-\rho}}; \quad \rho \left( \frac{\psi}{A} \right) = \rho$$

$$r \left( \frac{\psi}{A} \right) = \gamma \sigma^{\frac{\rho}{1-\rho}} \left[ 1 - \left( \frac{\psi}{\bar{z}A} \right)^{1-\rho} \right]^{\frac{\rho}{1-\rho}}; \quad \pi \left( \frac{\psi}{A} \right) = \gamma \sigma^{\frac{\rho}{1-\rho}} \left[ 1 - \left( \frac{\psi}{\bar{z}A} \right)^{1-\rho} \right]^{\frac{1}{1-\rho}}; \quad \ell \left( \frac{\psi}{A} \right) = \gamma \sigma^{\frac{\rho}{1-\rho}} \left( \frac{\psi}{\bar{z}A} \right)^{1-\rho} \left[ 1 - \left( \frac{\psi}{\bar{z}A} \right)^{1-\rho} \right]$$

with

- a constant pass-through rate, $0 < \rho < 1$.
- Employment hump-shaped with $\hat{z}/\bar{z} = (1 - \rho)^{1-\rho} > \hat{\psi}/\bar{z}A = (1 - \rho)^{1-\rho}$, both decreasing in $\rho$.
- CES is the limit case, as $\rho \to 1$, while holding $\beta > 0$ and $\sigma > 1$ fixed, so that $\sigma(\psi/A) \to \sigma; \quad \bar{z} \equiv \beta \left( \frac{\sigma}{\sigma - 1} \right)^{\frac{\rho}{1-\rho}} \to \infty$. 


\textbf{Power Elasticity of Markup Rate:} For $\kappa \geq 0$ and $\lambda > 0$

\[
s(z) = \exp \left[ \int_{z_0}^{z} \frac{c}{c - \exp \left[ -\frac{\kappa z^{-\lambda}}{\lambda} \right] \exp \left[ \frac{\kappa \xi^{-\lambda}}{\lambda} \right] \frac{d\xi}{\xi}} \right]
\]
with either $\bar{z} = \infty$ and $c \leq 1$ or $\bar{z} < \infty$ and $c = 1$. Then,

\[
1 - \frac{1}{\zeta(z)} = c \exp \left[ \frac{\kappa \bar{z}^{-\lambda}}{\lambda} \right] \exp \left[ -\frac{\kappa z^{-\lambda}}{\lambda} \right] < 1 \Rightarrow \mathcal{E}_{1-1}(z) = -\mathcal{E}_{\zeta/(\zeta-1)}(z) = \kappa z^{-\lambda}
\]
satisfying $A_2$ and the strong form of $A_3$ for $\kappa > 0$ and $\lambda > 0$.

CES for $\kappa = 0$; $\bar{z} = \infty$; $c = 1 - \frac{1}{\sigma}$; CoPaTh for $\bar{z} < \infty$; $c = 1$; $\kappa = \frac{1 - \rho}{\rho} > 0$, and $\lambda \to 0$.

- $\rho \left( \frac{\psi}{A} \right) = \frac{1}{1 + \kappa (z_\psi)^{-\lambda}}$, with $z_\psi = Z \left( \frac{\psi}{A} \right)$ given implicitly by $c \exp \left[ \frac{\kappa \bar{z}^{-\lambda}}{\lambda} \right] z_\psi \exp \left[ -\frac{\kappa (z_\psi)^{-\lambda}}{\lambda} \right] = \frac{\psi}{A}$.

- $\frac{\partial^2 \ln \rho(\psi/A)}{\partial A \partial \psi} \leq 0 \iff (\kappa)^{\frac{1}{\lambda}} \leq z_\psi = Z \left( \frac{\psi}{A} \right) \iff \frac{\psi}{A} \leq (\kappa)^{\frac{1}{\lambda}}c \exp \left[ \frac{\kappa \bar{z}^{-\lambda-1}}{\lambda} \right]$; Log-sub(super)modular among more (less) efficient firms. In particular, if $\bar{z} < (\kappa)^{\frac{1}{\lambda}}, \frac{\partial^2 \ln \rho(\psi/A)}{\partial A \partial \psi} < 0$ for all $\psi / A < Z(\psi/A) < \bar{z} < \infty$.

- Employment hump-shaped with the peak at $\hat{z} = Z \left( \frac{\psi}{A} \right) < \bar{z}$, given implicitly by

\[
c \left( 1 + \frac{\hat{z}^{\lambda}}{\kappa} \right) \exp \left[ -\frac{\kappa \hat{z}^{-\lambda}}{\lambda} \right] \exp \left[ \frac{\kappa \bar{z}^{-\lambda}}{\lambda} \right] = 1 \iff \left( 1 + \frac{\hat{z}^{\lambda}}{\kappa} \right) \hat{z} = \frac{\psi}{A}.
\]