Foundations for the New Keynesian Model

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Objective

• Describe a very simple model economy with no monetary frictions.
  – Describe its properties.
  – ‘markets work well’

• Modify the model to include price setting frictions.
  – Now markets won’t necessarily work so well, unless monetary policy is good.
Model

- Household preferences:

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1 + \varphi} \right\},
\]

\[
\tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \quad \varepsilon_t^\tau \sim iidN(0, \sigma^2_{\varepsilon})
\]
Production

• Final output requires lots of intermediate inputs:

\[ Y_t = \left[ \int_0^1 Y_{i,t}^{\varepsilon-1} \, di \right]^{\varepsilon-1}, \varepsilon > 1 \]

• Production of intermediate inputs:

\[ Y_{i,t} = e^{a_t} N_{i,t}, \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a, \varepsilon_t^a \sim iidN(0, \sigma_a^2) \]

• Constraint on allocation of labor:

\[ \int_0^1 N_{it} \, di = N_t \]
Efficient Allocation of Total Labor

• Suppose total labor, $N_t$, is fixed.

• What is the best way to allocate $N_t$ among the various activities, $0 \leq i \leq 1$?

• Answer:
  – allocate labor equally across all the activities

$$N_{it} = N_t, \text{ all } i$$
Suppose Labor Not Allocated Equally

• Example:

\[
N_{it} = \begin{cases} 
2\alpha N_t & i \in \left[0, \frac{1}{2}\right] \\
2(1-\alpha)N_t & i \in \left[\frac{1}{2}, 1\right]
\end{cases}, \quad 0 \leq \alpha \leq 1.
\]

• Note that this is a particular distribution of labor across activities:

\[
\int_0^1 N_{it} \, di = \frac{1}{2} 2\alpha N_t + \frac{1}{2} 2(1-\alpha)N_t = N_t
\]
Labor Not Allocated Equally, cnt’d

\[ Y_t = \left[ \int_0^1 Y_{i,t} \frac{\varepsilon-1}{\varepsilon-1} \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = \left[ \int_0^{\frac{1}{2}} Y_{i,t} \frac{\varepsilon-1}{\varepsilon} \, di + \int_{\frac{1}{2}}^1 Y_{i,t} \frac{\varepsilon-1}{\varepsilon} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{a_t} \left[ \int_0^{\frac{1}{2}} N_{i,t} \frac{\varepsilon-1}{\varepsilon} \, di + \int_{\frac{1}{2}}^1 N_{i,t} \frac{\varepsilon-1}{\varepsilon} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{a_t} \left[ \int_0^{\frac{1}{2}} (2\alpha N_t) \frac{\varepsilon-1}{\varepsilon} \, di + \int_{\frac{1}{2}}^1 (2(1 - \alpha) N_t) \frac{\varepsilon-1}{\varepsilon} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{a_t} N_t \left[ \int_0^{\frac{1}{2}} (2\alpha) \frac{\varepsilon-1}{\varepsilon} \, di + \int_{\frac{1}{2}}^1 (2(1 - \alpha)) \frac{\varepsilon-1}{\varepsilon} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{a_t} N_t \left[ \frac{1}{2} (2\alpha) \frac{\varepsilon-1}{\varepsilon} + \frac{1}{2} (2(1 - \alpha)) \frac{\varepsilon-1}{\varepsilon} \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{a_t} N_t f(\alpha) \]
$$f(\alpha) = \left[ \frac{1}{2} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2} (2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

Efficient Resource Allocation Means Equal Labor Across All Sectors

$\varepsilon = 10$

$\varepsilon = 6$
Economy with Efficient $N$ Allocation

- Efficiency dictates
  \[ N_{it} = N_t \text{ all } i \]

- So, with efficient production:
  \[ Y_t = e^{a_t} N_t \]

- Resource constraint:
  \[ C_t \leq Y_t \]

- Preferences:
  \[
  E_0 \sum_{t=0}^{\infty} \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \quad \tau_t = \lambda \tau_{t-1} + \varepsilon^t, \quad \varepsilon_t \sim \text{iid},
  \]
Efficient Determination of Labor

- Lagrangian:

\[
\max_{C_t, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} + \lambda_t [e^{a_t} N_t - C_t] \right\}
\]

- First order conditions:

\[
u_c(C_t, N_t, \tau_t) = \lambda_t, \quad u_n(C_t, N_t, \tau_t) + \lambda_t e^{a_t} = 0
\]

- or:

\[
u_{n,t} + e^{a_t} u_{c,t} = 0
\]

marginal cost of labor in consumption units = \frac{\frac{du}{dN_t}}{\frac{du}{dC_t}} = \frac{dC_t}{dN_t}

\[
\frac{-u_{n,t}}{u_{c,t}} = e^{a_t}
\]

marginal product of labor
Efficient Determination of Labor, cont’d

• Solving the fonc’s:

\[-u_{n,t} \quad u_{c,t} = e^{a_t}\]

\[C_t \exp(\tau_t) N_t^\phi = e^{a_t}\]

\[e^{a_t} N_t \exp(\tau_t) N_t^\phi = e^{a_t}\]

\[\rightarrow N_t = \exp\left( \frac{-\tau_t}{1 + \phi} \right)\]

\[\rightarrow C_t = \exp\left( a_t - \frac{\tau_t}{1 + \phi} \right)\]

• Note:

– Labor responds to preference shock, not to tech shock
Response to a Jump in $a$

![Diagram showing consumption, production frontier, higher $a$, and indifference curves.](image)
Decentralizing the Model

• Give households budget constraints and place them in markets.

• Give the production functions to firms and suppose that they seek to maximize profits.
Households

• Solve:

\[ \max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\phi}}{1 + \phi} \right\} , \]

• Subject to:

\[ C_t + \underbrace{B_{t+1}} \leq \underbrace{w_t N_t \, \pi_t} + \underbrace{r_{t-1} B_t} \]

bonds purchases in \( t \) \quad \text{wage rate} \quad \text{profits} \quad \text{(real) interest on bonds}

• First order conditions:

\[ \frac{-u_{n,t}}{u_{c,t}} = C_t \exp(\tau_t) N_t^\phi = w_t \quad \text{‘marginal cost of working equals marginal benefit’} \]

\[ u_{c,t} = \beta E_t u_{c,t+1} r_t \quad \text{‘marginal cost of saving equals marginal benefit’} \]
Final Good Firms

- Final good firms buy $Y_{i,t}, i \in (0,1)$, at given prices, $P_{i,t}$, to maximize profits:

$$Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

- Subject to

$$Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon - 1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon - 1}}$$

- Fonc’s:

$$P_{i,t} = \left( \frac{Y_t}{Y_{i,t}} \right)^{\frac{1}{\varepsilon}}$$

$$\rightarrow Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t, \ 1 = \int_0^1 P_{i,t}^{1-\varepsilon} di$$
Intermediate Good Firms

• For each $Y_{i,t}$ there is a single producer who is a monopolist in the product market and hires labor, $N_{i,t}$ in competitive labor markets.

• Marginal cost of production:

$$(\text{real) marginal cost} = s_t = \frac{d\text{Cost}}{d\text{worker}} = \frac{d\text{Cost}}{d\text{output}} = 1 - \frac{\text{subsidy payment to firm}}{\exp(a_t)} w_t$$

• Subsidy will be required to ensure markets work efficiently.
Intermediate Good Firms

\[ Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t \]
ith Intermediate Good Firm

- Problem: \( \max_{N_{it}} P_{it} Y_{it} - s_t Y_{it} \)

- Subject to demand for \( Y_{i,t} \): \( Y_{i,t} = P^{-\varepsilon}_{i,t} Y_t \)

- Problem:
  \[
  \max_{N_{it}} P_{it} P^{-\varepsilon}_{i,t} Y_t - s_t P^{-\varepsilon}_{i,t} Y_t
  \]

  \[\text{fonc : } (1 - \varepsilon)P^{-\varepsilon}_{it} Y_t + \varepsilon s_t P^{-\varepsilon - 1}_{i,t} Y_t = 0\]

  \[P_{it} = \frac{\varepsilon}{\varepsilon - 1} s_t \text{ ‘price is markup over marginal cost’}\]

- Note: all prices are the same, so resources allocated efficiently across intermediate good firms.

  \[P_{i,t} = P_{j,t} = 1, \text{ because } 1 = \int_0^1 P_{i,t}^{1-\varepsilon} di\]
Equilibrium

• Pulling things together:

\[
1 = \frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} \frac{(1 - \nu)w_t}{\exp(a_t)}
\]

= household function

\[
\frac{\varepsilon (1 - \nu)}{\varepsilon - 1} \frac{-u_{n,t}}{u_{c,t}} \frac{1}{\exp(a_t)}
\]

= if \( \frac{\varepsilon (1 - \nu)}{\varepsilon - 1} = 1 \)

If proper subsidy is provided to monopolists, employment is efficient:

\[
\text{if } 1 - \nu = \frac{\varepsilon - 1}{\varepsilon}, \text{ then } \frac{-u_{n,t}}{u_{c,t}} = \exp(a_t)
\]
Equilibrium Allocations

• With efficient subsidy,

\[ \frac{-u_{n,t}}{u_{c,t}} \xlongequal{\text{functional form}} C_t \exp(\tau_t) N_t^\phi \xlongequal{\text{resource constraint}} \exp(a_t) \exp(\tau_t) N_t^{1+\phi} = \exp(a_t) \]

\[ \Rightarrow N_t = \exp \left( \frac{-\tau_t}{1 + \phi} \right) \]

\[ C_t = e^{a_t} N_t = \exp \left( a_t - \frac{\tau_t}{1 + \phi} \right) \]

• Bond market clearing implies:

\[ B_t = 0 \text{ always} \]
Interest Rate in Equilibrium

- Interest rate backed out of household intertemporal Euler equation:

\[
u_{c,t} = \beta E_t u_{c,t+1} r_t \to \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} r_t
\]

\[
\rightarrow r_t = \frac{1}{\beta E_t \frac{c_t}{C_{t+1}}} = \frac{1}{\beta E_t \exp[c_t - c_{t+1}]} = \frac{1}{\beta E_t \exp[a_t - a_{t+1} - \frac{\tau_{t-\tau_{t+1}}}{1+\varphi}]}
\]

\[
= \frac{1}{\beta \exp[E_t(-\Delta a_{t+1} - \frac{\tau_{t-\tau_{t+1}}}{1+\varphi}) + \frac{1}{2} V]}, \quad V = \sigma^2_a + \left(\frac{1}{1 + \varphi}\right)^2 \sigma^2_\lambda
\]

\[
\log r_t = -\log \beta + E_t \left(\frac{c_{t+1} - c_t}{\Delta a_{t+1} - \frac{\tau_{t+1} - \tau_t}{1 + \varphi}}\right) + \frac{1}{2} V
\]
Interest Rate in Equilibrium

• Interest rate backed out of household intertemporal Euler equation:

\[ u_{c,t} = \beta E_t u_{c,t+1} r_t \rightarrow \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} r_t \]

\[ \rightarrow r_t = \frac{1}{\beta E_t \frac{C_t}{C_{t+1}}} = \frac{1}{\beta E_t \exp[c_t - c_{t+1}]} = \frac{1}{\beta E_t \exp[a_t - a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\phi}]} \]

\[ = \frac{1}{\beta \exp\left[ E_t \left( -\Delta a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\phi} \right) + \frac{1}{2} V \right]}, \quad V = \sigma_a^2 + \left( \frac{1}{1 + \phi} \right)^2 \sigma_\lambda^2 \]

\[ \log r_t = -\log \beta + E_t \left( \Delta a_{t+1} - \frac{\tau_{t+1} - \tau_t}{1 + \phi} \right) + \frac{1}{2} V \]

using assumptions about \( \Delta a_t \) and \( \tau_t \)

\[ \approx -\log \beta + \rho \Delta a_t - \frac{(\lambda-1)\tau_t}{1+\phi} + \frac{1}{2} V \]
Dynamic Properties of the Model

Response to .01 Technology Shock in Period 1

\[(a_t - a_{t-1}) = 0.75(a_{t-1} - a_{t-2}) + \varepsilon_{a_t}\]

Response to .01 Preference Shock in Period 1

\[\tau_t = 0.5\tau_{t-1} + \varepsilon_{\tau_t}\]

Interest rate

\[\log r_t = -\log \beta + 0.75\Delta a_t\]

\[\log r_t = -\log \beta - (0.5 - 1)\tau_t/(1 + 1)\]
Key Features of Equilibrium Allocations

• Allocations *efficient* (as long as monopoly power neutralized)

• Employment does not respond to technology
  – Improvement in technology raises marginal product of labor and marginal cost of labor by same amount.

• First best consumption not a function of intertemporal considerations
  – Discount rate irrelevant.
  – Anticipated future values of shocks irrelevant.

• Natural rate of interest steers consumption and employment towards their natural levels.
Introducing Price Setting Frictions (Clarida-Gali-Gertler Model)

- Households maximize:

\[ E_0 \sum_{t=0}^{\infty} \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \quad \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \quad \varepsilon_t^\tau \sim iid, \]

- Subject to:

\[ P_tC_t + B_{t+1} \leq W_t N_t + R_{t-1} B_t + T_t \]

- Intratemporal first order condition:

\[ C_t \exp(\tau_t) N_t^\varphi = \frac{W_t}{P_t} \]
Household Intertemporal FONC

• Condition:

\[ 1 = \beta E_t \frac{u_{c,t+1}}{u_{c,t}} \frac{R_t}{1 + \pi_{t+1}} \]

– or

\[ 1 = \beta E_t \frac{C_t}{C_{t+1}} \frac{R_t}{1 + \pi_{t+1}} \approx \beta \exp[\log(R_t) - \log(1 + \pi_{t+1}) - \Delta c_{t+1}] \]

\[ \simeq \beta \exp[\log(R_t) - E_t \pi_{t+1} - E_t \Delta c_{t+1}], \quad c_t \equiv \log(C_t) \]

– take log of both sides:

\[ 0 = \log(\beta) + r_t - E_t \pi_{t+1} - E_t \Delta c_{t+1}, \quad r_t = \log(R_t) \]

– or

\[ c_t = -\log(\beta) - [r_t - E_t \pi_{t+1}] + c_{t+1} \]
Final Good Firms

• Buy $Y_{i,t}, i \in [0, 1]$ at prices $P_{i,t}$ and sell $Y_t$ for $P_t$
• Take all prices as given (competitive)
• Profits:
  
  $$P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

• Production function:
  
  $$Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon}{\varepsilon-1}} di \right]^{\frac{\varepsilon-1}{\varepsilon}}, \varepsilon > 1,$$

• First order condition:
  
  $$Y_{i,t} = Y_t \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \quad \rightarrow \quad P_t = \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}$$
Intermediate Good Firms

• Each ith good produced by a single monopoly producer.

• Demand curve:

\[ Y_{i,t} = Y_t \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \]

• Technology:

\[ Y_{i,t} = \exp(a_t)N_{i,t}, \quad \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a, \]

• Calvo Price-setting Friction

\[ P_{i,t} = \begin{cases} \tilde{P}_t \text{ with probability } 1 - \theta \\ P_{i,t} \text{ with probability } \theta \end{cases}, \]
real marginal cost = \[ S_t = \frac{d\text{Cost}}{d\text{worker}} = \frac{d\text{Output}}{d\text{worker}} = \frac{(1 - \nu)W_t/P_t}{\exp(a_t)} \]

= \frac{\nu-1}{\nu} \text{ in efficient setting}

= \frac{(1 - \nu)C_t \exp(\tau_t)N_t}{\exp(a_t)}
The Intermediate Firm’s Decisions

• *ith* firm is required to satisfy whatever demand shows up at its posted price.

• Its only real decision is to adjust price whenever the opportunity arises.
Intermediate Good Firm

• Present discounted value of firm profits:

\[ E_t \sum_{j=0}^{\infty} \beta^j \left( \frac{v_{t+j}}{\beta^j} \right) \]

- marginal value of dividends to household = \(u_{c,t+j}/P_{t+j}\)
- period \(t+j\) profits sent to household

\[ \left\{ \begin{align*}
& \text{revenues} \\
& P_{i,t+j}Y_{i,t+j} - P_{t+j}s_{t+j}Y_{i,t+j}
\end{align*} \right\} \]

- total cost

• Each of the \(1 - \theta\) firms that can optimize price choose \(\tilde{P}_t\) to optimize

\[ E_t \sum_{j=0}^{\infty} \beta^j \left( \frac{\theta^j}{\beta^j} \right) \]

in selecting price, firm only cares about future states in which it can’t reoptimize

\[ v_{t+j}[\tilde{P}_tY_{i,t+j} - P_{t+j}s_{t+j}Y_{i,t+j}] \]
Intermediate Good Firm Problem

• Substitute out the demand curve:

\[
E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} \left[ \tilde{P}_t Y_{t,t+j} - P_{t+j} s_{t+j} Y_{t,t+j} \right]
\]

\[
= E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} Y_{t+j} P_t^\varepsilon \left[ (1 - \varepsilon) (\tilde{P}_t)^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_t^{-1} \right] = 0,
\]

• Differentiate with respect to \( \tilde{P}_t \):

\[
E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} Y_{t+j} P_t^\varepsilon \left[ (1 - \varepsilon) (\tilde{P}_t)^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_t^{-1} \right] = 0,
\]

• or

\[
E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} Y_{t+j} P_t^{\varepsilon+1} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.
\]
Intermediate Good Firm Problem

- Objective:

\[
E_t \sum_{j=0}^{\infty} (\beta \theta)^j \frac{u'(C_{t+j})}{P_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.
\]

\[
\rightarrow E_t \sum_{j=0}^{\infty} (\beta \theta)^j P_{t+j}^{\varepsilon} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.
\]

\[
E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{-\varepsilon} \left[ \tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0,
\]

\[
\tilde{p}_t = \frac{\tilde{P}_t}{P_t}, \quad X_{t,j} = \begin{cases} 
\frac{1}{\pi_{t+j} \pi_{t+j-1} \cdots \pi_{t+1}}, & j \geq 1 \\
1, & j = 0.
\end{cases}
\]

, \quad X_{t,j} = X_{t+1,j-1} \frac{1}{\pi_{t+1}}, \quad j > 0
Intermediate Good Firm Problem

• Want $\tilde{p}_t$ in:

$$
E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{-\varepsilon} \left[ \tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0
$$

• Solution:

$$
\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}
$$

• But, still need expressions for $K_t, F_t$. 
\[ K_t = E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \sum_{j=1}^{\infty} (\beta \theta)^{j-1} \left( \frac{1}{\pi_{t+1}} X_{t+1,j-1} \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta \theta)^j X_{t+1,j} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \]

\[ = E_t \text{ by LIME} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta \overline{E_t E_{t+1}} \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta \theta)^j X_{t+1,j} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} E_{t+1} \sum_{j=0}^{\infty} (\beta \theta)^j X_{t+1,j} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} K_{t+1} \]

exactly \( K_{t+1} \)!
From previous slide:

\[ K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} K_{t+1}. \]

Substituting out for marginal cost:

\[ \frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} (1 - \nu) \left( \frac{W_t}{P_t} \right) \frac{\text{exp}(\tau_t)N^\phi_t C_t}{\text{exp}(a_t)} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} (1 - \nu) \frac{\text{exp}(\tau_t)N^\phi_t C_t}{\text{exp}(a_t)} \] by household optimization.
In Sum

• solution:

\[
\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t},
\]

• Where:

\[
K_t = (1 - \nu_t) \frac{\varepsilon}{\varepsilon - 1} \frac{\exp(\tau_t)N_t^\phi C_t}{\exp(a_t)} + \beta \theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}.
\]

\[
F_t \equiv E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{1-\varepsilon} = 1 + \beta \theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{1-\varepsilon} F_{t+1}
\]
To Characterize Equilibrium

• Have equations characterizing optimization by firms and households.

• Still need:
  – Expression for all the prices. Prices, $P_{i,t}$, $0 \leq i \leq 1$, will all be different because of the price setting frictions.
  – Relationship between aggregate employment and aggregate output not simple because of price distortions:

\[ Y_t \neq e^{a_t}N_t, \text{ in general} \]

• This part of the analysis is the reason why it made Calvo famous – it’s not easy.
Aggregate Price Index

- Rewrite the aggregate price index.
  - let $p \in (0, \infty)$ the set of logically possible prices for intermediate good producers.
  - let $g_t (p) \geq 0$ denote the measure (e.g., ‘number’) of producers that have price, $p$, in $t$
  - let $g_{t-1,t} (p) \geq 0$, denote the measure of producers that had price, $p$, in $t - 1$ and could not reoptimize in $t$

- Then,

$$P_t = \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{-\frac{1}{1-\varepsilon}} = \left( \int_0^\infty g_t (p) p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}.$$

- Note:

$$P_t = \left( (1 - \theta) \tilde{P}_t^{1-\varepsilon} + \int_0^\infty g_{t-1,t} (p) p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}.$$
Aggregate Price Index

- Calvo randomization assumption:

\[
\left\{ \begin{array}{l}
g_{t-1,t}(p) \\
= \theta \times g_{t-1}(p)
\end{array} \right.
\]

measure of firms that had price, \( p \), in \( t-1 \) and could not change

measure of firms that had price \( p \) in \( t-1 \)

- Then,

\[
P_t = \left( (1 - \theta) \tilde{P}_{t}^{1-\varepsilon} + \int_{0}^{\infty} g_{t-1,t}(p) p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}
\]

\[
= \left( (1 - \theta) \tilde{P}_{t}^{1-\varepsilon} + \theta \int_{0}^{\infty} g_{t-1}(p) p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}
\]

\[
= P_{t-1}^{1-\varepsilon}
\]
Expression for $\tilde{P}_t$ in terms of aggregate inflation

- Conclude that this relationship holds between prices:

$$P_t = \left[ (1 - \theta)\tilde{P}_t^{(1-\varepsilon)} + \theta P_{t-1}^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}.$$

  – Only two variables here!

- Divide by $P_t$:

$$1 = \left[ (1 - \theta)\tilde{P}_t^{(1-\varepsilon)} + \theta \left( \frac{1}{\pi_t} \right)^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}$$

- Rearrange:

$$\tilde{P}_t = \left[ \frac{1 - \theta \pi_t^{(\varepsilon-1)}}{1 - \theta} \right]^{\frac{1}{1-\varepsilon}}$$
Relation Between Aggregate Output and Aggregate Inputs

- Technically, there is no ‘aggregate production function’ in this model.
  - If you know how many people are working, \( N \), and the state of technology, \( a \), you don’t have enough information to know what \( Y \) is.
  - Price frictions imply that resources will not be efficiently allocated among different inputs.
    - Implies \( Y \) low for given \( a \) and \( N \). How low?
    - Tak Yun (JME) gave a simple answer.
Tak Yun Algebra

\[ Y^*_t = \int_0^1 Y_{i,t} \, di \left( \equiv \int_0^1 A_t N_{i,t} \, di \right) \]

[labour market clearing]

\[ Y_t \int_0^1 \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \, di \]

\[ = Y_t P_t^\varepsilon \int_0^1 (P_{i,t})^{-\varepsilon} \, di \]

\[ = Y_t P_t^\varepsilon (P^*_t)^{-\varepsilon} \]

- **Where:**

\[ P_t^* \equiv \left[ \int_0^1 P_{i,t}^{-\varepsilon} \, di \right]^{-\frac{1}{\varepsilon}} = \left[ (1 - \theta) \tilde{P}_t^{-\varepsilon} + \theta (P^*_t)^{-\varepsilon} \right]^{-\frac{1}{\varepsilon}} \]
Relationship Between Agg Inputs and Agg Output

- Rewriting previous equation:

\[ Y_t = \left( \frac{P_t^*}{P_{t}} \right)^\varepsilon Y_t^* \]

\[ = p_t^* e^{a_t N_t}, \]

- ‘efficiency distortion’:

\[ p_t^* : \begin{cases} 
\leq 1 \\
= 1 \quad P_{i,t} = P_{j,t}, \text{ all } i,j 
\end{cases} \]
Let \( f(x) = x^4 \), a convex function. Then,

\[
\text{convexity: } \alpha x_1^4 + (1 - \alpha) x_2^4 > (\alpha x_1 + (1 - \alpha) x_2)^4
\]

for \( x_1 \neq x_2, 0 < \alpha < 1 \).

Applying this idea to prices:

\[
\text{convexity: } \int_0^1 \left( P_{i,t}^{(1-\varepsilon)} \right)^{\frac{\varepsilon}{\varepsilon - 1}} \, di \geq \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} \, di \right)^{\frac{\varepsilon}{\varepsilon - 1}}
\]

\[
\iff \left( \int_0^1 P_{i,t}^{\varepsilon} \, di \right) \geq \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} \, di \right)^{\frac{\varepsilon}{\varepsilon - 1}}
\]

\[
\iff \left( \int_0^1 P_{i,t}^{-\varepsilon} \, di \right)^{\frac{1}{\varepsilon}} \leq \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} \, di \right)^{\frac{1}{1-\varepsilon}}
\]
Example of Efficiency Distortion

\[
P_{j,t} = \begin{cases} 
P^1 & 0 \leq j \leq \alpha \\
P^2 & \alpha \leq j \leq 1 \\
\end{cases} \quad p_t^* = \left( \frac{p_t^*}{P_t} \right)^\varepsilon = \left( \frac{\alpha + (1 - \alpha)\left( \frac{p^2}{P^1} \right)^{-\varepsilon} }{\left[ \alpha + (1 - \alpha)\left( \frac{p^2}{P^1} \right)^{1-\varepsilon} \right]^\frac{1}{1-\varepsilon}} \right)^\varepsilon
\]
Example of Efficiency Distortion

\[ P_{j,t} = \begin{cases} P^1 & 0 \leq j \leq \alpha \\ P^2 & \alpha \leq j \leq 1 \end{cases} \]

\[ p_t^* = \left( \frac{P_t^*}{P_t} \right)^\varepsilon = \left( \frac{\alpha + (1 - \alpha) \left( \frac{p^2}{p^1} \right)^{-\varepsilon}}{\left[ \alpha + (1 - \alpha) \left( \frac{p^2}{p^1} \right)^{1-\varepsilon} \right]^\frac{1}{1-\varepsilon}} \right)^\varepsilon \]

\[ \alpha = 0.5, \varepsilon = 10 \]

\[ \log \frac{P^1}{P^2} \]
Collecting Equilibrium Conditions

- Price setting:

\[ K_t = (1 - \nu) \frac{\varepsilon}{\varepsilon - 1} \frac{\exp(\tau_t)N_t^\phi C_t}{A_t} + \beta \theta E_t \bar{\pi}_t^{\varepsilon} K_{t+1} \quad (1) \]

\[ F_t = 1 + \beta \theta E_t \bar{\pi}_t^{\varepsilon-1} F_{t+1} \quad (2) \]

- Intermediate good firm optimality and restriction across prices:

\[ \frac{\hat{K}_t}{\hat{F}_t} = \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \left( \frac{1}{1-\varepsilon} \right) \quad (3) \]
Equilibrium Conditions

- Law of motion of (Tak Yun) distortion:

\[ p_t^* = \left( 1 - \theta \right) \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right) \frac{\varepsilon}{\varepsilon-1} + \frac{\theta \bar{\pi}_t^\varepsilon}{p_{t-1}^*} \]  \hspace{1cm} (4)

- Household Intertemporal Condition:

\[ \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \]  \hspace{1cm} (5)

- Aggregate inputs and output:

\[ C_t = p_t^* e^{a_t} N_t \]  \hspace{1cm} (6)

- 6 equations, 8 unknowns:

\( \nu, C_t, p_t^*, N_t, \bar{\pi}_t, K_t, F_t, R_t \)

- System under determined!
Underdetermined System

• Not surprising: we added a variable, the nominal rate of interest.

• Also, we’re counting subsidy as among the unknowns.

• Have two extra policy variables.

• One way to pin them down: compute optimal policy.
Ramsey-Optimal Policy

• 6 equations in 8 unknowns…..
  – Many configurations of the 8 unknowns that satisfy the 6 equations.
  – Look for the best configurations (Ramsey optimal)
    • Value of tax subsidy and of $R$ represent optimal policy

• Finding the Ramsey optimal setting of the 6 variables involves solving a simple Lagrangian optimization problem.
Ramsey Problem

$$\max_{v, p^*_t, C_t, N_t, R_t, \pi_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1 + \varphi} \right)$$

$$+ \lambda_{1t} \left[ \frac{1}{C_t} - E_t \frac{\beta}{C_{t+1}} \frac{R_t}{\pi_{t+1}} \right]$$

$$+ \lambda_{2t} \left[ \frac{1}{p^*_t} - \left( 1 - \theta \right) \left( \frac{1 - \theta(\pi_t)^{\varepsilon-1}}{1 - \theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \pi_t^\varepsilon}{p^*_{t-1}} \right]$$

$$+ \lambda_{3t} \left[ 1 + E_t \pi_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t \right]$$

$$+ \lambda_{4t} \left[ (1 - \nu) \frac{\varepsilon}{\varepsilon - 1} \frac{C_t \exp(\tau_t) N_t^\varphi}{e^{\alpha_t}} + E_t \beta \theta \pi_{t+1}^\varepsilon K_{t+1} - K_t \right]$$

$$+ \lambda_{5t} \left[ F_t \left( \frac{1 - \theta \pi_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right]$$

$$+ \lambda_{6t} \left[ C_t - p_t^* e^{\alpha_t} N_t \right] \}$$
Solving the Ramsey Problem (surprisingly easy in this case)

- First, substitute out consumption everywhere

\[
\max_{v, p^*_t, N_t, R_t, \bar{\pi}_t, F_t, K_t} \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log N_t + \log p^*_t - \exp(\tau_t) \frac{N_t^{1+\phi}}{1+\varphi} \right) \right. \\
+ \lambda_{1t} \left[ \frac{1}{p^*_t N_t} - E_t \frac{e^{a_t} \beta}{p^*_t e^{a_{t+1}} N_{t+1} \bar{\pi}_{t+1}} R_t \right] \\
+ \lambda_{2t} \left[ \frac{1}{p^*_t} - \left( (1 - \theta) \left( \frac{1 - \theta(\bar{\pi}_t)^{\varepsilon-1}}{1 - \theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \theta \bar{\pi}_t^\varepsilon \right) \right] \\
+ \lambda_{3t} [1 + E_t \bar{\pi}_t^{\varepsilon-1} \beta \theta F_{t+1} - F_t] \\
+ \lambda_{4t} \left[ (1 - v) \frac{\varepsilon}{\varepsilon - 1} \exp(\tau_t) N_t^{1+\phi} p^*_t + E_t \beta \theta \bar{\pi}_t^{\varepsilon} K_{t+1} - K_t \right] \\
+ \lambda_{5t} \left[ F_t \left( \frac{1 - \theta \bar{\pi}_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \left\} \right.
\]
Solving the Ramsey Problem (surprisingly easy in this case)

• First, substitute out consumption everywhere

\[
\max_{v, p_t^*, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1 + \varphi} \right) \right\}
\]

defines \( R \)

\[
+ \lambda_{1_t} \left[ \frac{1}{p_t^* N_t} - E_t \frac{e^{\alpha_t} \beta}{p_{t+1}^* e^{\alpha_{t+1}} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]
\]

defines \( F \)

\[
+ \lambda_{3_t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t]
\]

defines tax

\[
+ \lambda_{4_t} \left[ (1 - v) \frac{\varepsilon}{\varepsilon - 1} \exp(\tau_t) N_t^{1+\varphi} p_t^* + E_t \beta \theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right]
\]

defines \( K \)

\[
+ \lambda_{5_t} \left[ F_t \left( \frac{1 - \theta \bar{\pi}_t^{\varepsilon-1}}{1 - \theta} \right) \frac{1}{1-\varepsilon} - K_t \right] \}
Solving the Ramsey Problem, cnt’d

• Simplified problem:

\[ \max_{\bar{\pi}_t, p_t^*, N_t} \mathbb{E}_{0} \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\phi}}{1+\phi} \right) \right\} \]

\[ + \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( 1 - \theta \right) \left( \frac{1 - \theta(\bar{\pi}_t)^{\varepsilon-1}}{1 - \theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^\varepsilon}{p_{t-1}^*} \right] \}

• First order conditions with respect to \( p_t^*, \bar{\pi}_t, N_t \)

\[ p_t^* + \beta \lambda_{2, t+1} \theta \bar{\pi}_{t+1} = \lambda_{2t}, \quad \bar{\pi}_t = \left[ \frac{(p_{t-1})^{\varepsilon-1}}{1 - \theta + \theta(p_{t-1})^{\varepsilon-1}} \right]^{\frac{1}{\varepsilon-1}}, \quad N_t = \exp\left( -\frac{\tau_t}{\phi + 1} \right) \]

• Substituting the solution for inflation into law of motion for price distortion:

\[ p_t^* = \left[ (1 - \theta) + \theta(p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}. \]
Eventually, price distortions are eliminated, regardless of shocks. The solution to the Ramsey Problem is given by:

$$p_t^* = \left( (1 - \theta) + \theta(p_{t-1}^*)^{(\varepsilon-1)} \right)^{1/(\varepsilon-1)}$$

When price distortions are gone, so is inflation:

$$\pi_t = \frac{p_{t-1}^*}{p_t^*}$$

Efficient (‘first best’) allocations in the real economy:

$$N_t = \exp\left(-\frac{\tau_t}{1 + \varphi}\right)$$

$$1 - \nu = \frac{\varepsilon - 1}{\varepsilon}$$

$$C_t = p_t^* e^{a_t} N_t.$$
Eventually, Optimal (Ramsey) Equilibrium and Efficient Allocations in Real Economy Coincide

Convergence of price distortion

\[ \theta = 0.75, \ \varepsilon = 10 \]

\[ p_i^* = \left[ (1 - \theta) + \theta(p_{i-1}^*)^{(e-1)} \right]^{\frac{1}{(e-1)}} \]
• The Ramsey allocations are eventually the best allocations in the economy without price frictions (i.e., ‘first best allocations’)

• Refer to the Ramsey allocations as the ‘natural allocations’....
  – Natural consumption, natural rate of interest, etc.