Simple New Keynesian Model without Capital: Benchmarks

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Target Equilibrium

• Want to evaluate performance of NK model with Taylor rule.
• How well does Taylor rule work? How to improve it?
• For this, must have in mind a target, benchmark, to evaluate economic performance.
• Important indicator of performance,

\[
\text{output gap} = \text{actual} - \text{desired output}
\]

what is ‘desired output’?
Benchmarks for NK Equilibrium

• One benchmark: *Natural Equilibrium*, $\theta = 0$.
  – Convenient, but not sure what it means to shut down sticky prices (and wages).
  – These are typically modeled in reduced form.
    • Presumably reflect something real in the economy. What does it mean to assume those things away?

• Alternative benchmark: *Ramsey Equilibrium*.
  – Ramsey equilibrium: *best possible* equilibrium, taking frictions in the economy as given.
  – Problem: sometimes hard to compute (in principle, Dynare does it).

• Simplify things: *cashless limit*.
  – In the simple model and in the cashless limit: Natural and Ramsey equilibrium coincide.
Equilibrium Conditions, NK Model

- 9 equations in 9 unknowns: $m_t, C_t, p^*_t, F_t, K_t, N_t, R_t, \bar{\pi}_t, \mu_t$ and 2 exogenous variables: $\nu, G_t$.

$K_t = \frac{\varepsilon}{\varepsilon - 1} \frac{Y_t}{C_t} \frac{(1 - \nu) e^{\bar{\pi}_t C_t N_t}}{e^{a_t}} + \beta \theta E_t \bar{\pi}_{t+1} K_{t+1}$ (1)

$F_t = \frac{Y_t}{C_t} + \beta \theta E_t \bar{\pi}_{t+1} F_{t+1}$ (2), $\frac{K_t}{F_t} = \left[ 1 - \theta \bar{\pi}_t^{(\varepsilon - 1)} \right] \frac{1}{1 - \varepsilon}$ (3)

$p^*_t = \left[ (1 - \theta) \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon - 1)}}{1 - \theta} \right) \frac{\varepsilon}{\varepsilon - 1} + \frac{\theta \bar{\pi}_t^\varepsilon}{p^*_t - 1} \right]^{-1}$ (4)

$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}}$ (5), $C_t + G_t = p^*_t e^{a_t N_t}$ (6)

$m_t = \frac{\gamma C_t}{\left( 1 - \frac{1}{R_t} \right)}$ (7), $m_t = \left( \frac{1 + \mu_t}{\bar{\pi}_t} \right) m_{t-1}$ (8)

$R_t / R = \left( R_{t-1} / R \right)^\alpha \exp \left[ (1 - \alpha) \phi \pi (\bar{\pi}_t - \tilde{\pi}) + \phi x x_t \right]$ (7)'.

Natural Equilibrium

- When $\theta = 0$, then

$$
\frac{\varepsilon (1 - \nu)}{\varepsilon - 1} \times e^{\tau t} C_t N_t^\phi = e^{\alpha t}
$$

so that we have a form of efficiency when $\nu$ is chosen to that
$\varepsilon (1 - \nu) / (\varepsilon - 1) = 1$.

- In addition, recall that we have allocative efficiency in the flexible price equilibrium.

- So, the flexible price equilibrium with the efficient setting of $\nu$ represents a natural benchmark for the New Keynesian model, the version of the model in which $\theta > 0$.
  - We call this the Natural Equilibrium.

- To simplify the analysis, from here on we set $G_t = 0$. 

Natural Equilibrium

- With $G_t = 0$, equilibrium conditions for $C_t$ and $N_t$:

  
  \[
  \frac{e^{\tau_t} C_t N_t^\varphi}{e^{a_t}} = \exp\left(\frac{-\tau_t}{1 + \varphi}\right)
  \]

  aggregate production relation: $C_t = e^{a_t} N_t$.

- Substituting,

  \[
  e^{\tau_t} e^{a_t} N_t^{\varphi + 1} = e^{a_t} \rightarrow N_t = \exp\left(\frac{-\tau_t}{1 + \varphi}\right)
  \]

  \[
  C_t = \exp\left(a_t - \frac{\tau_t}{1 + \varphi}\right)
  \]

  \[
  R_t^* = \frac{1}{\beta E_t C_t} = \frac{1}{\beta E_t C_{t+1}} = \frac{1}{\beta E_t \exp\left(-\Delta a_{t+1} + \frac{\Delta \tau_{t+1}}{1 + \varphi}\right)}
  \]
Natural Equilibrium, cnt’d

- Natural rate of interest:

\[ R^*_t = \frac{1}{C_t} \frac{1}{\beta E_t C_{t+1}} = \frac{1}{\beta E_t \exp \left( -\Delta a_{t+1} + \frac{\Delta \tau_{t+1}}{1+\varphi} \right)} \]

- Two models for \( a_t \):

  \( DS \) : \( \Delta a_{t+1} = \rho \Delta a_t + \varepsilon^{a}_{t+1} \)
  \( TS \) : \( a_{t+1} = \rho a_t + \varepsilon^{a}_{t+1} \)

- Model for \( \tau_t \):

  \( \tau_{t+1} = \lambda \tau_t + \varepsilon^{\tau}_{t+1} \)
Natural Equilibrium, cnt’d

• Suppose the $\varepsilon_t$’s are Normal. Then,

$$E_t \exp \left( -\Delta a_{t+1} + \frac{\Delta \tau_{t+1}}{1 + \varphi} \right) = \exp \left( -E_t \Delta a_{t+1} + E_t \frac{\Delta \tau_{t+1}}{1 + \varphi} + \frac{1}{2} V \right)$$

where

$$V = \sigma_a^2 + \frac{\sigma^2_\tau}{(1 + \varphi)^2}$$

• Then, with $r^*_t \equiv \log R^*_t$

$$r^*_t = -\log \beta + E_t \Delta a_{t+1} - E_t \frac{\Delta \tau_{t+1}}{1 + \varphi} - \frac{1}{2} V.$$

• Useful: consider how natural rate responds to $\varepsilon^a_t$ shocks under DS and TS models for $a_t$ and how it responds to $\varepsilon^\tau_t$ shocks.
  – To understand how $r^*_t$ responds, consider implications of consumption smoothing in absence of change in $r^*_t$.
  – In natural equilibrium, $r^*_t$ steers the economy so that natural equilibrium paths for $C_t$ and $N_t$ are realized.
Ramsey Equilibrium

• Problem with Natural (‘Flexible Price’) Equilibrium
  – Not clear what we are doing when we set θ = 0.
  – Source of price frictions is not specified.

• An alternative concept is the Ramsey Equilibrium
  – Drop Taylor rule
  – Replace $C_t$ by $p^*_t \exp (a_t) N_t$ everywhere.
  – So, have 7 equations in 9 unknowns:
    \[ m_t, p^*_t, F_t, K_t, N_t, R_t, \pi_t, \mu_t, \nu. \]
  – System is now underdetermined (2 equations short).

• Ramsey equilibrium:
  – Values of (‘stochastic processes for’) variables that yield
    highest social welfare (or, utility of monetary policy authority)
  – Constrained efficient.
  – Ramsey Equilibrium is solution to Ramsey Problem.
Ramsey Problem

\[
\max_{\nu, \mu, m_t, p_t^*, F_t, K_t, N_t, R_t, \pi_t} \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log (N_t p_t^*) - \frac{\exp (\tau_t) N_t^{1+\varphi}}{1 + \varphi} + \gamma \log (m_t) \right) \right.
\]

\[
+ \lambda_{1,t} \left[ \frac{1}{p_t^* N_t} - \beta E_t \frac{e^{a_t}}{p_{t+1}^* N_{t+1} e^{a_{t+1}} \pi_{t+1}} \frac{R_t}{\pi_{t+1}} \right]
\]

\[
+ \lambda_{2,t} \left[ \frac{1}{p_t^*} - \left( 1 - \theta \right) \left( \frac{1 - \theta \pi_t^{(\varepsilon-1)}}{1 - \theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \pi_t^{\varepsilon}}{p_{t-1}^*} \right]
\]

\[
+ \lambda_{3,t} \left[ 1 + \beta \theta E_t \pi_{t+1}^{\varepsilon-1} F_{t+1} - F_t \right] + \lambda_{7,t} \left[ m_t - \left( \frac{1 + \mu_t}{\pi_t} \right) m_{t-1} \right]
\]

\[
+ \lambda_{4,t} \left[ \frac{\varepsilon}{\varepsilon-1} (1 - \nu) e^{\tau_t} p_t^* N_t^{1+\varphi} + \beta \theta E_t \pi_{t+1}^{\varepsilon} K_{t+1} - K_t \right]
\]

\[
+ \lambda_{5,t} \left[ K_t - F_t \left( \frac{1 - \theta \pi_t^{(\varepsilon-1)}}{1 - \theta} \right)^{\frac{1}{1-\varepsilon}} \right] + \lambda_{6,t} \left[ m_t - \frac{R_t}{R_t - 1} \gamma p_t^* N_t e^{a_t} \right] \}
\]
Cashless Limit

- Recall preferences:

\[ E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log (N_t) + \log (p_t^*) - \frac{\exp(\tau_t) N_t^{1+\phi}}{1 + \phi} + \gamma \log (m_t) \right\} \]

where \( \gamma = 0.0051 \).

- Solution to Ramsey problem roughly the same if we set \( \gamma = 0 \) and ignore equations:

\[ m_t = \frac{R_t}{R_t - 1} \gamma p_t^* N_t e^{at} \]

\[ m_t = \left( \frac{1 + \mu_t}{\pi_t} \right) m_{t-1} \]

- Appears generally true that if we ignore money demand and supply, properties of model not substantially affected.
  
  - Woodford (2003).
Ramsey Problem in Cashless Limit

\[
\max_{\nu, \mu, p_t^*, F_t, K_t, N_t, R_t, \bar{\pi}_t} \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log(N_t) + \log(p_t^*) - \frac{\exp(\tau_t) N_t^{1+\varphi}}{1 + \varphi} \right) + \lambda_1, t \left[ \frac{1}{p_t^* N_t} - \beta E_t \frac{e^{a_t}}{p_{t+1}^* N_{t+1} e^{a_{t+1}}} \frac{R_t}{\bar{\pi}_{t+1}} \right] + \lambda_2, t \left[ \frac{1}{p_t^*} - \left( 1 - \theta \right) \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right) \frac{\varepsilon}{\varepsilon - 1} + \theta \bar{\pi}_t^{\varepsilon} \right] + \lambda_3, t \left[ 1 + \beta \theta E_t \bar{\pi}_t^{\varepsilon-1} F_{t+1} - F_t \right] + \lambda_4, t \left[ \frac{\varepsilon}{\varepsilon - 1} \left( 1 - \nu \right) e^{\tau_t} p_t^* N_t^{1+\varphi} + \beta \theta E_t \bar{\pi}_t^{\varepsilon} K_{t+1} - K_t \right] + \lambda_5, t \left[ K_t - F_t \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right) \frac{1}{1 - \varepsilon} \right] \right\}
\]
A Note on Lagrange Multipliers

• Constrained optimization problem \((y > 0, \text{exogenous})\):

\[
\max_c \log(c) \\
\text{subject to: } c \leq y, c \geq 0.
\]

• Lagrangian representation of problem:

\[
\max L = \log(c) + \lambda [y - c] + \mu c 
\]

where the last multiplier captures constraint, \(c \geq 0\).

• Multipliers: ‘prices’ associated with constraints:
  – Price of budget constraint is positive, \(\lambda > 0\), because with \(\lambda = 0\) solution would be \(c = \infty\)
    • budget constraint limits how big the objective can be.
  – Price of constraint, \(c \geq 0\), is zero because if you set \(\mu = 0\) and solve the resulting Lagrangian, you satisfy \(c \geq 0\) automatically
    • non-negativity constraint on consumption does not limit how big the objective can be (constraint is non-binding).
Ramsey Problem in Cashless Limit

\[
\max_{\nu, \mu_t, p_t^*, F_t, K_t, N_t, R_t, \bar{\pi}_t} \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log (N_t) + \log (p_t^*) - \frac{\exp (\tau_t) N_t^{1+\varphi}}{1 + \varphi} \right) \right. \\
+ \lambda_{1,t} \left[ \frac{1}{p_t^* N_t} - \beta E_t p_t^* e^{\eta_t} \frac{e^{\eta_t}}{p_t^{*+1} N_{t+1} e^{\eta_{t+1}}} \frac{R_t}{\bar{\pi}_{t+1}} \right] \\
+ \lambda_{2,t} \left[ \frac{1}{p_t^*} - \left( 1 - \theta \right) \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right) \frac{\varepsilon}{\varepsilon-1} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_t^{*-1}} \right] \\
+ \lambda_{3,t} \left[ 1 + \beta \theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1} - F_t \right] \\
+ \lambda_{4,t} \left[ \frac{\varepsilon}{\varepsilon-1} (1 - \nu) e^{\tau_t} p_t^* N_t^{1+\varphi} + \beta \theta E_t \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right] \\
\left. + \lambda_{5,t} \left[ K_t - F_t \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right) \frac{1}{1-\varepsilon} \right] \right\}
\]

- **Conjecture:**
  - Can use (1) to define \( R_t \) without affecting anything else, so \( \lambda_{1,t} = 0 \)
  - Can use (3),(4),(5) to define \( F_t, K_t, \nu \), so \( \lambda_{3,t} = \lambda_{4,t} = \lambda_{5,t} = 0 \).
Ramsey Problem in Cashless Limit

• Simplified problem under conjecture:

$$\max_{\bar{\pi}_t, p_t^*, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log N_t + \log p_t^* - \exp \left( \tau_t \frac{N_t^{1+\varphi}}{1+\varphi} \right) \right\}$$

$$+ \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( 1 - \theta \right) \left( \frac{1 - \theta (\bar{\pi}_t)^{\varepsilon-1}}{1 - \theta} \right) \frac{\varepsilon}{\varepsilon-1} + \frac{\theta \bar{\pi}_t^\varepsilon}{p_t^*} \right]$$

first order conditions with respect to $p_t^*$, $\bar{\pi}_t$, $N_t$ (after rearranging):

$$p_t^* + \beta \lambda_{2,t+1} \theta \bar{\pi}_{t+1}^\varepsilon = \lambda_{2t},$$

$$\bar{\pi}_t = \left[ \frac{(p_{t-1}^*)^{\varepsilon-1}}{1 - \theta + \theta (p_{t-1}^*)^{\varepsilon-1}} \right] \frac{1}{\varepsilon-1}, \quad N_t = \exp \left( -\frac{\tau_t}{\varphi+1} \right)$$

– Substituting the solution for $\bar{\pi}_t$ into the law of motion for $p_t^*$:

$$p_t^* = \left[ (1 - \theta) + \theta (p_{t-1}^*)^{\varepsilon-1} \right]^{\frac{1}{(\varepsilon-1)}}, \quad \bar{\pi}_t = \frac{p_{t-1}^*}{p_t^*}.$$
The Other Constraints: Intertemporal Euler Equation

• Choose $R_t$ so the intertemporal constraint is satisfied:

$$R_t = \frac{1}{p_t^* N_t} \frac{1}{p_t^* e^{(a_t-a_{t+1}) \beta} E_t \frac{p_t^*}{p_{t+1} N_{t+1} \bar{\pi}_{t+1}}} = \frac{1}{e^{a_t N_t}} \frac{1}{\beta E_t e^{a_{t+1} N_{t+1}}},$$

using

$$\bar{\pi}_t = \frac{p_t^*}{p_t^*}.\]$$

• We have ignored the zero lower bound on $R_t$.
  - If that is violated because the shocks are large, then the fact that we implicitly set the multiplier on the ZLB to zero would be an error.
Price Setting Equations

- Price setting conditions:

\[ 1 + E_t \bar{\pi}_{t+1}^{\epsilon-1} \beta \theta F_{t+1} = F_t \quad (1) \]

\[ (1 - \nu_t) \frac{\epsilon}{\epsilon - 1} \exp(\tau_t) N_t^{1+\varphi} p_t^* + E_t \bar{\pi}_t^\epsilon \beta \theta K_{t+1} = K_t \quad (2) \]

\[ F_t \left[ \frac{1 - \theta \bar{\pi}_t^{\epsilon-1}}{1 - \theta} \right]^{1/\epsilon} \text{(making use of the expression for optimal inflation)} \]

\[ \frac{F_t}{p_t^*} = K_t \quad (3) \]

* Divide (2) by \( p_t^* \), impose (3) and use \( \bar{\pi}_{t+1} = p_t^*/p_{t+1}^* \):

\[ \frac{\epsilon}{\epsilon - 1} \times (1 - \nu) \times \exp(\tau_t) N_t^{1+\varphi} + E_t \bar{\pi}_t^{\epsilon-1} \beta \theta F_{t+1} = F_t \]

- Subtract from (1) (subsidy must cancel markup and interest rate distortion):

\[ (1 - \nu) \frac{\epsilon}{\epsilon - 1} = 1. \]
Bottom Line

• Solution to Ramsey problem in cashless limit:

\[ p_t^* = \left[ (1 - \theta) + \theta \left( p_{t-1}^* \right)^{(\varepsilon-1)} \right] \frac{1}{(\varepsilon-1)} \]

\[ \bar{\pi}_t = \frac{p_{t-1}^*}{p_t^*}, \quad N_t = \exp \left( -\frac{\tau_t}{1+\varphi} \right) \]

\[ 1 - \nu_t = \frac{\varepsilon - 1}{\varepsilon}, \quad C_t = p_t^* e^{a_t} N_t, \quad R_t = \frac{1}{\beta E_t \frac{1}{e^{a_t} e^{a_{t+1}} N_{t+1}}} \]

• Properties of Ramsey
  – Eventually, same as natural equilibrium with zero inflation.
  – Ramsey-optimal policy is time consistent
    • Two tools (monetary policy and \( \nu \)) to solve two problems: sticky prices and monopoly power.

• Role of Cashless Limit Assumption
  – with \( \gamma > 0 \) but small, results would be similar, but optimal inflation would be slightly negative.
    • “Cost of price dispersion beats Friedman rule”.
Thoughts About the Cashless Limit

• Monetary models make no sense unless there is a demand and supply of money.
  – Central banks affect the interest rate through their control of the quantity of money (e.g., bank reserves).
  – That control would not be possible absent a demand for money.

• There are two cases to consider.
  – Case 1: real balances enter the model additively separably, e.g., in the utility function separably from other variables like consumption and labor.
    • This is what I assumed in my discussion.
  – Case 2: real balances do not enter separably.
    • This is the only reasonable assumption, since we think that money is desired for its usefulness in making transactions.
Cashless Limit: Case 1

Case 1: real balances enter the model additively separably, e.g., in the utility function separably from other variables like consumption and labor.

- In this case, saw that if you have an interest rate rule and want to solve, simulate and estimate the model, no need to think about money.
- Also true if want to compute optimal policy (e.g., Norges Bank, Riksbank) using monetary policy committee objective function, if it does not include money balances (‘modified inflation targeting’).
- If you want to do Ramsey, then money matters mathematically.
  - But, numerical results roughly the same if you ignore money.
  - In our example, that’s because the empirically relevant value of $\gamma$ is small (0.0052), because velocity is high (e.g., around 2.2).
  - When you set $\gamma = 0$ it’s not because you think there is no demand for money. It’s just a numerical approximation that simplifies the calculations.
Cashless Limit: Case 2

Case 2: real balances do not enter separably.

- This is the economically interesting case, since money is used for transactions.
  - For example, rather than having $\log \left[ \frac{M_{t+1}}{P_t} \right]$ in the utility function it makes more sense to have $\log \left[ \frac{M_{t+1}}{(P_tC_t)} \right]$.

- The non-separable case has been considered.
  - Numerical simulations have tended to support the following proposition:
    - ‘As long as you parameterize the model so that velocity corresponds to what we observe in the data: ignoring money demand or including it yields numerically similar results.’
  - See Woodford (2003) for more discussion.
  - The proposition now permeates most of the New Keynesian analyses.
  - Important to understand and re-examine this surprising proposition.