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Learning, convergence, and stability with multiple rational expectations equilibria*

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For a linear model with multiple rational expectations equilibria (REE) we discuss the set of possible REE. Solutions include (i) locally unique minimal state variable solutions, taking an AR(1) form and (ii) continua of solutions which may depend on sunspots. We analyze the convergence of econometric learning to the different REE. There exist cases with more than one stable AR(1) solution. It is also possible for a continuum of solutions to be stable, but this property is not robust to overparametrization. An application is developed, and it is suggested that the exhibited phenomena may arise in applied macroeconomic models.

Key words: Sunspots; Learning; Multiplicity; Expectational stability

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1. Introduction

The Rational Expectations (RE) hypothesis is pervasive in economics and is now the assumption of choice when building macroeconomic models. A practical problem in RE modelling is that a model can have multiple equilibria. Indeed, multiple formal solutions will typically arise in dynamic expectations models, including even fully specified general equilibrium models. Such concepts as fundamental solutions, bubbles and sunspots have

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been used to describe various rational expectations equilibria (REE).[1] With multiple equilibria the predictive power of the RE hypothesis alone is weak, and additional criteria for finding 'reasonable' solutions are needed. Various selection criteria have been proposed in the literature including stationarity of equilibria, minimum variance solutions, minimal state variable solutions, and expectational stability of equilibria (see e.g. McCallum (1983) and Evans and Honkapohja (1992a) for discussions and references).

Stability under adaptive learning – the object of study in this paper – has also been increasingly adopted as a useful selection criterion.[2] In learning behavior agents are boundedly rational during the adjustment process. They are assumed to follow a natural statistical learning rule which has the potential to converge to an REE. Much of the early literature analyzed convergence of learning rules in models with a unique REE, but some recent papers have considered convergence of learning with multiple equilibria. A reasonable solution is then an equilibrium that is a locally stable outcome of such learning processes.

The focus of this paper is the role of learning in linear dynamic expectations models with multiple REE. We examine this case, because linear RE models are the standard tool for applied macroeconomists. One purpose of the paper is to show that straightforward approaches are available for both the derivation of the set of possible REE and the analysis of convergence of econometric learning rules for linear models.

In the simple linear models that have been heretofore examined, there have been no cases of multiple REE which are robustly stable under learning.[3] This raises the question: is this a general result? The second purpose of this paper is to show that the answer is negative. Multiple strongly stable solutions (in the sense defined below) can exist for more general models.

Our basic approach can be summarised as follows. For concreteness consider the model

$$y_t = \alpha + \delta y_{t-1} + \sum_{i=0}^{m} \beta_i ({}_{t-1}y^e_{t+i}) + v_t, \tag{1.1}$$

where y_t is a scalar endogenous variable, ${}_{t-1}y^e_{t+i}$ denotes expectations of y_{t+i} formed at the end of period $t-1$, and v_t is a sequence of independent

[1] See e.g. McCallum (1983), Evans and Honkapohja (1986), Broze et al. (1987), and Chiappori and Guesnerie (1988) for results, examples and further references.

[2] For convergence of least squares learning see Bray (1982), Bray and Savin (1986), Fourgeaud et al. (1986), Marcet and Sargent (1988, 1989a, b), Woodford (1990), and Evans and Honkapohja (1992b, c). Related approaches include, inter alia, Grandmont and Laroque (1986, 1991) and Guesnerie and Woodford (1991).

[3] The case $m=1$ for (1.1) below is the most general linear model analyzed comprehensively so far. Evans and Honkapohja (1992a) examined this case of $m=1$ for E-stability. The current paper sets $m=2$ and establishes the connection between econometric learning and E-stability.

disturbances. For applied macroeconomists the standard solution procedure would be to obtain what are often called minimal state variable solutions, see McCallum (1983). For model (1.1) this would amount to assuming a first-order autoregressive (AR(1)) form at the outset:

$$y_t = a + b_1 y_{t-1} + v_t. \tag{1.2}$$

This is a common practice and is appropriate for avoiding bubble or sunspot REE. It is known that in general there can exist $m+1$ different solutions of form (1.2) to model (1.1).[4] In addition to these solutions (which are locally unique in the parameter space), various solution continua to (1.1) also exist. These continua may involve *sunspot solutions*, i.e. solutions which depend on extraneous random variables through expectations formation. We show in section 2 (and Appendix A) how the method of undetermined coefficients can be used to obtain the full set of solutions.

For learning dynamics we establish and exploit the connection between stability under learning and a disequilibrium stability criterion known as expectational stability (E-stability).[5] We show that this connection, established by Marcet and Sargent (1988, 1989a) in models with a unique REE, holds also for local stability in models with multiple REE. The basic result for the AR(1) solutions above is that such an REE is stable under learning if and only if it is E-stable. (Analogous but more subtle results hold for the solutions continua.)

The concept of E-stability can be easily illustrated as follows for the case $m=2$. Suppose agents believe that the economy follows an AR(1) process (1.2). In specifying this *perceived law of motion* we do not impose rational expectations, though we have chosen a form which corresponds to one class of RE solutions. We next calculate the expectation functions that agents would have, given these perceptions. The corresponding expectation functions are:

$$_{t-1}y_t^e = a + b_1 y_{t-1},$$

$$_{t-1}y_{t+1}^e = a + b_1(_{t-1}y_t^e) = a(1+b_1) + b_1^2 y_{t-1},$$

$$_{t-1}y_{t+2}^e = a + b_1(_{t-1}y_{t+1}^e) = a(1+b_1+b_1^2) + b_1^3 y_{t-1}.$$

Inserting these expectations into (1.1) with $m=2$ we obtain the *actual law of motion* generated by these perceptions:

[4]They can be obtained by computing the expectations $E_{t-1}y_{t+i}$ from (1.2), inserting into (1.1) and solving the resulting $m+1$ degree polynomial. There are fewer than $m+1$ solutions of form (1.2) if some roots of the polynomial are nonreal.

[5]E-stability has, for example, been considered in Lucas (1978, section 6), DeCanio (1979), Bray (1982, Proposition 4), Evans (1985, 1986, 1989) and Evans and Honkapohja (1992a).

$$y_t = T_a(a, b_1) + T_{b_1}(a, b_1)y_{t-1} + v_t, \quad \text{where}$$

$$T_a(a, b_1) = \alpha + a((\beta_0 + \beta_2) + (b_1 + 1)(\beta_1 + \beta_2 b_1)), \tag{1.3a}$$

$$T_{b_1}(a, b_1) = \delta + \beta_0 b_1 + \beta_1 b_1^2 + \beta_2 b_1^3. \tag{1.3b}$$

This defines a mapping $T(a, b_1) = (T_a(a, b_1), T_{b_1}(a, b_1))'$ from the perceived to the actual law of motion. (′ denotes the transpose.) Note that the AR(1) REE correspond to the fixed points of T.

E-stability is then defined in terms of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(a, b_1)' = T(a, b_1) - (a, b_1)'. \tag{1.4}$$

This equation describes a stylized learning process in notional time τ, in which the perceived law of motion $(a, b_1)'$ is partially adjusted towards the actual law $T(a, b_1)$ generated by the perceptions.[6] An AR(1) solution which is locally stable in terms of this differential equation is said to be (weakly) *E-stable*. The results in section 3 will give the conditions for E-stability, while section 4 establishes the connection with adaptive learning.

Although formal proofs of convergence under econometric learning are technical, and rest on the stochastic approximation literature, the intuitive reason for the connection with E-stability is straightforward. Under econometric learning, once the sample size is sufficiently large, each new observation generates a forecast error which on average moves the estimated parameters of the perceived law of motion a small increment toward the actual law. This adjustment is approximated by (1.4), with the link between notional time τ and real time t approximately given, for t large, by $\tau(t) - \tau(t-1) = t^{-1}$.

In this paper we will analyze comprehensively the model (1.1) with $m = 2$. This is a simple and special case, but it is sufficient for showing the existence of multiple stable solutions, and for exhibiting the approach and techniques which can be applied in more general contexts. An open economy macro example fitting exactly the case $m = 2$ is discussed in section 5.

For this model there is a domain of parameter values for which there is a unique REE that is stable under learning. However, it is also possible to find other parameter regions which yield multiple robustly stable AR(1) solutions. This suggests that this possibility may arise in applied high order linear

[6] E-stability is sometimes defined in terms of iterations on the T mapping, and the concept of iterative E-stability can be useful in discussions of 'educative' justifications of RE. However, it is the differential equation formulation which has precise connections to convergence of learning algorithms, and is therefore the concept employed here. For a brief comparison, see Evans (1989).

macroeconomic models. In addition, we show that for some parameter regions econometric learning can converge to continua of solutions depending on sunspots, though this result is very sensitive to the form of the learning rule. Practitioners should be aware of these phenomena and the tools for analyzing the stability of the various REE under learning.

2. The formal framework

The analysis is carried out using the following class of scalar linear models:

$$y_t = \alpha + \delta y_{t-1} + \beta_0({}_{t-1}y_t^e) + \beta_1({}_{t-1}y_{t+1}^e) + \beta_2({}_{t-1}y_{t+2}^e) + v_t, \tag{2.1}$$

where $\beta_2 \neq 0$. The disturbance term v_t satisfies $\mathrm{E}_{t-1}v_{t+s} = 0$ $\forall s \geq 0$. ${}_{t-1}y_{t+s}^e$ denotes the expectation formed at the end of period $t-1$ which agents hold for the value of y_{t+s}. Under rational expectations we have ${}_{t-1}y_{t+s}^e = \mathrm{E}_{t-1}y_{t+s}$, where $\mathrm{E}_{t-1}()$ denotes the mathematical expectation conditional on information at $t-1$. The information set is the usual one, consisting of past values of the endogenous variable and the disturbance term as well as of any sunspot process that might influence the economy through expectations. Note that the information set does not include current values of the variables.

(2.1) is not a general formulation, but the earlier literature, as noted above, has looked at even more specific cases. (2.1) provides a simple linear model which can have multiple stable equilibria and solution continua of different dimensionalities.

2.1. The RE solutions

The simple method mentioned in the introduction can be used to obtain minimal state variable solutions. For a comprehensive listing of all REE, however, it is more convenient to view these REE as special cases of the general representation, see Broze et al. (1987) and Evans and Honkapohja (1986).

This general representation is carried out in terms of finite ARMA processes in the endogenous variable y_t and the disturbance term v_t, together with a moving average of an arbitrary martingale difference sequence w_t. w_t can either represent an extraneous (sunspot) process or can depend upon v_t. (We may assume without loss of generality that w_t is not a linear function of the process v_t.) For convenience, we will refer to w_t as a sunspot. Note that, although we will refer to ARMA or AR solutions, v_t and w_t need not in general have constant variances. Appendix A derives the general form of the solutions with sunspots.

The set of REE for model (2.1) can be written as follows:

(i) The ARMA representation of highest AR degree is ARMA(3, 2) in variables y_t and v_t plus a MA(2) process in terms of w_t:

$$y_t = -\alpha\beta_2^{-1} - \beta_1\beta_2^{-1}y_{t-1} + (1-\beta_0)\beta_2^{-1}y_{t-2} - \delta\beta_2^{-1}y_{t-3} + v_t + c_1v_{t-1} + c_2v_{t-2} + d_1w_{t-1} + d_2w_{t-2}. \quad (2.2)$$

The AR coefficients in (2.2) can be obtained from (2.1) by replacing ${}_{t-1}y^e_{t+s}$ by y_{t+s} and multiplying by β_2^{-1}. The coefficients for the moving average terms c_1, c_2, d_1, and d_2 are all arbitrary, as is the sunspot process itself. (It is easy to verify by substitution that (2.2) solves (2.1).) We will refer to (2.2) as the *ARMA*(3, 2) *class of equilibria.* (2.2) can also be written in terms of polynomials of the lag operator L in the factored form

$$(1-\rho_1L)(1-\rho_2L)(1-\rho_3L)y_t = -\beta_2^{-1}\alpha + (1-\mu_1L)(1-\mu_2L)v_t + L(d_1+d_2L)w_t. \quad (2.2')$$

The μ_i's in (2.2′) can clearly take any value. In writing (2.2′) we have, for clarity of exposition, assumed that the left-hand side has three real roots. (If not, it can be factored into a first-degree and a quadratic polynomial.) The relationship between ρ_i's and the parameters of the model is

$$\beta_2^{-1}\beta_1 = -(\rho_1+\rho_2+\rho_3), \quad \beta_2^{-1}(\beta_0-1) = \rho_1\rho_2+\rho_1\rho_3+\rho_2\rho_3,$$
$$\beta_2^{-1}\delta = -\rho_1\rho_2\rho_3. \quad (2.3)$$

(ii) There exist, generically, one or three classes of equilibria of ARMA(2, 1) type in y_t and v_t plus a possible sunspot term. These are obtained by eliminating one of the AR(1) roots. Choosing, for example, $\mu_1=\rho_3$ and $d_2=-\rho_3d_1$ we have the common factor $(1-\rho_3L)$ on both sides of (2.2′). Cancelling it we obtain

$$(1-\rho_1L)(1-\rho_2L)y_t = -\alpha\beta_2^{-1}(1-\rho_3)^{-1} + v_t + \xi v_{t-1} + \kappa w_{t-1}, \quad (2.4)$$

where ξ and κ are arbitrary constants.[7] We will refer to these as *ARMA*(2, 1) *classes of equilibria.* The corresponding AR parameters on y_{t-1} and y_{t-2} are $\bar{b}_1=(\rho_1+\rho_2)$, $\bar{b}_2=-\rho_1\rho_2$.

(iii) There exist, generically, one or three REE which are AR(1) processes in (y_t, v_t), with no dependence on a sunspot process. These 'minimal state variable solutions' are the standard REE referred to above, and are obtained by eliminating two common factors in (2.2′) without the sunspot terms (i.e. $d_1=d_2=0$). Cancelling e.g. the factors $(1-\rho_2L)$ and $(1-\rho_3L)$ gives

[7]Elimination of common factors also requires an appropriate choice of initial conditions, see Evans and Honkapohja (1986).

$$(1-\rho_1 L)y_t = -\alpha\beta_2^{-1}(1-\rho_2)^{-1}(1-\rho_3)^{-1} + v_t. \tag{2.5}$$

Subsequently, these will be called (ρ_i) *AR*(1) *equilibria*, where ρ_i stands for the remaining root on the left-hand side.

2.2. Expectational stability

The general concept of E-stability for the various REE is formulated along the lines outlined in the introductory discussion of section 1. Consider perceived laws of motion of the form

$$y_t = a + \sum_{i=1}^{s} b_i y_{t-i} + v_t + \sum_{i=1}^{r} c_i v_{t-i} + \sum_{i=1}^{q} d_i w_{t-i}. \tag{2.6}$$

Note that for appropriate s, r, and q (2.6) nests all of the REE. As before we compute ${}_{t-1}y^e_{t+i}$, for $i=0,1,2$, as

$${}_{t-1}y^e_t = a + \sum_{i=1}^{s} b_i y_{t-i} + \sum_{i=1}^{r} c_i v_{t-i} + \sum_{i=1}^{q} d_i w_{t-i},$$

$${}_{t-1}y^e_{t+1} = a + b_1({}_{t-1}y^e_t) + \sum_{i=2}^{s} b_i y_{t+1-i} + \sum_{i=2}^{r} c_i v_{t+1-i} + \sum_{i=2}^{q} d_i w_{t+1-i},$$

$${}_{t-1}y^e_{t+2} = a + b_1({}_{t-1}y^e_{t+1}) + b_2({}_{t-1}y^e_t) + \sum_{i=3}^{s} b_i y_{t+2-i} + \sum_{i=3}^{r} c_i v_{t+2-i} + \sum_{i=3}^{q} d_i w_{t+2-i}.$$

If s, r, or $q \leqslant 2$, the appropriate terms in the above expression are set equal to 0.

Inserting these expressions into (2.1) we obtain an actual law of motion of the same form as (2.6). This defines a mapping $\theta^* = T(\theta)$, where $\theta' = (a, b', c', d')$ and we have introduced the vector notation $b' = (b_1, \ldots, b_s)$, $c' = (c_1, \ldots, c_r)$ and $d' = (d_1, \ldots, d_q)$. (Again $'$ denotes the transpose.) The explicit T-mapping is given in the beginning of Appendix B.

E-stability is then defined in terms of the differential equation

$$\mathrm{d}\theta/\mathrm{d}\tau = T(\theta) - \theta. \tag{2.7}$$

It will be important to make a distinction between weak and strong E-stability. An AR(1) equilibrium is said to be *strongly E-stable* if (2.7) is locally stable at that solution for every choice of $s, r, q \geqslant 1$. For *weak E-stability* of the AR(1) solution we require only local stability of the subsystem in the variables (a, b_1) with b_i for $i \geqslant 2$, and c and d set identically to 0. A set of ARMA(2, 1) solutions with sunspots is weakly E-stable, if there is local convergence of (2.7) to that set for $s=2$, $r=1$ and $q=1$. For strong E-stability we require local convergence for every $s \geqslant 2$, $r \geqslant 1$ and $q \geqslant 1$. (For ARMA(2, 1) solutions without sunspots one omits the term $\sum_{i=1}^{q} d_i w_{t-i}$.) For the set of ARMA(3, 2) equilibria these concepts are defined analogously.

Weak E-stability permits only perturbations in the coefficients of the variables that enter the REE being considered, while strong E-stability requires robustness also with respect to overparametrizations.

3. E-stability of rational expectations equilibria

3.1. AR(1)-equilibria

We begin with the REE that can be represented as AR(1) processes in y_t with disturbance term v_t (and no dependence on sunspots). As noted above, depending on the values of the structural parameters δ, β_0, β_1, and β_2 there generically exist one or three such equilibria. Without loss of generality, we focus on the AR(1) equilibrium with root ρ_1:[8]

Proposition 3.1. The ρ_1-AR(1) equilibrium is weakly E-stable if (a) $\beta_2(\rho_1-\rho_2)(\rho_1-\rho_3)<0$, and (b) $\beta_2(1-\rho_2)(1-\rho_3)<0$. It is strongly E-stable, if additionally (c) $\beta_2\rho_2\rho_3<0$.

It is necessary to recall relations (2.3) when applying the stability conditions.[9] In many cases there is a unique strongly E-stable AR(1) equilibrium. However, manipulating conditions (a)–(c) we obtain:

Corollary 1. Suppose there are three AR(1) solutions and without loss of generality assume $\rho_1<\rho_2<\rho_3$. If the ρ_2-AR(1) equilibrium is weakly E-stable, then the other AR(1) solutions are not. The ρ_1-AR(1) and ρ_3-AR(1) equilibria are both strongly E-stable if (a) $\beta_2<0$, and (b) either $1<\rho_1<\rho_2<\rho_3$ or $0<\rho_1<\rho_2<\rho_3<1$ or $\rho_1<\rho_2<\rho_3<0$.

This possibility of multiple strongly E-stable solutions in linear models has not been previously noticed. In section 5 we illustrate this and other possibilities in an economic model that fits our framework. Intuitively, the possibility of multiple E-stable solutions is evident from (1.4) since the differential equation for b_1 is a cubic and it is independent of the other variable a.

There is no simple connection between E-stability and stochastic stationar-

[8]The proofs of E-stability results are given in Appendix B.

[9]There is a subtlety which arises when $\alpha=0$: Strictly speaking, weak E-stability should in that case be defined using only the differential equation for b_1. Then condition (a) is alone sufficient. An analogous point arises for the ARMA solutions when the constant term is absent.

ity, even in the 'saddle-point stable' case in which the model has a unique stationary equilibrium. In particular, suppose the model satisfies $-1<\rho_1<0<1<\rho_2<\rho_3$ and $\beta_2>0$. Then the ρ_1-AR(1) solution is uniquely stationary, whereas the ρ_2-AR(1) equilibrium is uniquely strongly E-stable. This shows that in general E-stability leads to selection of equilibria which are distinct from those that would be given by other proposed criteria. Evans and Honkapohja (1992a) discuss at some length the relationships between the various selection criteria (when $\beta_2=0$).

The following result provides some additional understanding of the above phenomena:

Corollary 2. Let $-1<\delta<1$ be given. For all β_0, β_1 and β_2 sufficiently small in magnitude, there is a unique stationary AR(1) solution and it is uniquely strongly E-stable.

Thus the possibilities of multiple E-stable AR(1) solutions and of nonstationary E-stable solutions can only arise (for $-1<\delta<1$) when there is a sufficiently strong dependence of the current state on the expected state of the economy[10].

3.2. ARMA-equilibria

Next we discuss the stability properties of the other possible REE to model (2.1). Recall from section 2 that these take the form of ARMA(3, 2) and, generically, one or three ARMA(2, 1) continua, which may or may not depend on sunspot variables. They were illustrated by formulae (2.2) and (2.4) respectively. The different classes of ARMA solutions can sometimes be weakly E-stable:

Proposition 3.2. There exists a nontrivial domain of the parameters $(\beta_0, \beta_1, \beta_2, \delta)$ such that the class of ARMA(3, 2) equilibria is weakly E-stable.

Proposition 3.3. There exists a nontrivial domain of the parameters $(\beta_0, \beta_1, \beta_2, \delta)$ such that a class of ARMA(2, 1) equilibria is weakly E-stable.

Appendix B gives the precise stability conditions for each case.[11] A numerical example will be given in section 4. However, it turns out that the ARMA solutions are fragile in the following sense:

Proposition 3.4. The classes of ARMA solutions are never strongly E-stable.

[10]Note that under the assumptions of Corollary 2 the ARMA solutions are nonstationary. It is shown below that the ARMA solutions are never strongly E-stable. It can also be verified that there are domains of parameter values for which (i) there is a unique stationary AR(1) solution and it is strongly E-stable and (ii) all other solutions fail to be even weakly E-stable.

[11]Note that an ARMA class without sunspots is weakly E-stable under the same conditions as the corresponding ARMA class with sunspots.

Remark. The lack of strong stability is delicate. If an ARMA solution set is weakly E-stable, the overparametrized system (2.7) will exhibit one-sided stability/instability, with convergence from some nearby starting points and divergence from others.

In summary, the ARMA continua of REE, with or without sunspots, can at best be only weakly E-stable. If strong E-stability is desired, one must limit attention to the sunspot-free AR(1) equilibria which are locally unique in the parameter space (Evans and Honkapohja, 1986, p. 232).[12]

In Section 4 we exhibit simulations showing convergence and nonconvergence to a weakly E-stable continuum. There are two reasons for devoting some attention to such solutions. First, because of the substantial current interest in sunspot solutions as a possible explanation of macroeconomic fluctuations, we feel it is important to fully investigate their stability properties under learning. Second, we note that there are examples of models in which there are weakly E-stable solutions, but no strongly E-stable solutions (Evans and Honkapohja, 1992a, Corollary 2, p. 7; Evans, 1989, Section 4). Thus, in some cases, insisting on strong E-stability may be too stringent.

4. Real time learning

We now take up the issue of the evolution of the system under real time learning rules. At time $t-1$ agents are assumed to have the perceived law of motion

$$y_t = a_{t-1} + \sum_{i=1}^{s} b_{i,t-1} y_{t-i} + \sum_{i=1}^{r} c_{i,t-1} v_{t-i} + v_t + \sum_{i=1}^{q} d_{i,t-1} w_{t-i} \tag{4.1}$$

which they use to form subjective expectations ${}_{t-1}y^e_{t+j}$, $j=0,1,2$[13]. We assume that a_{t-1}, $b'_{t-1}=(b_{1,t-1},\dots,b_{s,t-1})$, $c'_{t-1}=(c_{1,t-1},\dots,c_{r,t-1})$ and $d'_{t-1}=(d_{1,t-1},\dots,d_{q,t-1})$ are updated each period using a recursive econometric algorithm, as described below. The value of y_t is then given by (2.1). We are interested in finding which rational expectations equilibria can be possible long-run outcomes and in the conditions for local convergence. We argue that the answers are determined by the E-stability conditions derived in the preceding section. Throughout this section we assume that v_t and w_t are mutually independent sequences of identically and independently distributed random variables with 0 means and bounded moments.

[12]This gives some support for McCallum's (1983) primary selection criterion, based on the minimal state variables principle. However, concerning his subsidiary principle see Evans and Honkapohja (1992a) and the discussion after Corollary 1 to Proposition 3.1 above.

[13]This assumes that the exogenous shock v_t is in the information set I_t. If it remains unobservable then v_t itself must be estimated. See below.

When demonstrating convergence of econometric learning algorithms our proofs rely on the 'ordinary differential equation approach' for analysing stochastic recursive systems developed by Ljung (1977) and applied to RE models by Marcet and Sargent (1989a, b).

A preliminary comment is in order. Since our results are proved using Marcet and Sargent (1989a), our algorithm incorporates a 'projection facility' which ensures that estimates do not leave some bounded set containing the REE parameters under consideration. The use of a projection facility has recently been criticized by Grandmont and Laroque (1991). We regard the projection facility as a convenient and appropriate technical device for examining the *local* stability of particular equilibria in the context of a general model with multiple solutions. Its role is to eliminate the possibility that random shocks, at an early stage of the process, displace estimates outside the domain of attraction of the REE equilibrium being considered. Using a projection facility one obtains a very strong sense of stochastic convergence, almost sure convergence, when the E-stability condition is met. Weaker notions of stochastic convergence are available for algorithms without the projection facility at the cost of considerable technical complication, see Evans and Honkapohja (1994).[14] In fact, in most of the simulations below we dispense with the projection facility.

4.1. AR(1) case

We begin with the AR(1) case in which the perceived law of motion is given by[15] $y_t = a_{t-1} + b_{t-1} y_{t-1} + v_t$, so that the actual law of motion is

$$y_t = T_a(a_{t-1}, b_{t-1}) + T_{b_1}(a_{t-1}, b_{t-1}) y_{t-1} + v_t, \tag{4.2}$$

where $T_a(.)$ and $T_{b_1}(.)$ are given by (1.3a) and (1.3b), respectively. Below we write $T(a, b) = (T_a(a, b), T_{b_1}(a, b))'$.

As remarked below, the algorithm to update (a_t, b_t) is in fact a modification of recursive least squares estimation. Our formulation will also be suitable for estimation of higher order processes. Let

$$\theta_t' = (a_t, b_t), \tag{4.3a}$$

$$z_{t-1}' = (1, y_{t-1}), \tag{4.3b}$$

and define $(\bar{\theta}_t, \bar{R}_t)$ by

[14] We also note that in particular (nonlinear) models it has been possible to obtain theoretical results of almost sure convergence without the use of a projection facility, see e.g. Woodford (1990), and Evans and Honkapohja (1992b).

[15] One could also allow for the possibility that agents are estimating the coefficient of the v_t shock.

$$\bar{\theta}_t = \theta_{t-1} + (\alpha_t/t)(\bar{R}_t)^{-1} z_{t-1}\epsilon_t, \tag{4.4a}$$

$$\bar{R}_t = R_{t-1} + (\alpha_t/t)(z_{t-1}z_{t-1}' - R_{t-1}/\alpha_t), \quad \text{where} \tag{4.4b}$$

$$\epsilon_t = y_t - z_{t-1}'\theta_{t-1}, \tag{4.4c}$$

$$y_t = z_{t-1}'T(\theta_{t-1}) + v_t, \tag{4.4d}$$

and the weights α_t form a positive, nondecreasing sequence with $\alpha_t \to 1$ as $t \to \infty$ and $\limsup t|\alpha_t - \alpha_{t-1}| = K < \infty$ as $t \to \infty$. For later convenience we have rewritten (4.2), the actual law of motion for y_t, as (4.4d) and included it in the algorithm equations.

A *projection facility* is defined by $D_2 \subset D_1 \subset \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}^n)$, with $n=2$ here. We assume that D_1 is open and bounded and that D_2 is a closed set of positive measure. D_2 must be such that its projection onto $\mathbb{R}^n$ contains the REE of interest in its interior. The perceived law of motion evolves according to the algorithm

$$(\theta_t, R_t) = \begin{cases} (\bar{\theta}_t, \bar{R}_t) & \text{if} \quad (\bar{\theta}_t, \bar{R}_t) \in D_1 \\ \text{some value in } D_2 & \text{if} \quad (\bar{\theta}_t, \bar{R}_t) \notin D_1. \end{cases} \tag{4.4e}$$

Remark. For $D_2 = D_1 = \mathbb{R}^2 \times (\mathbb{R}^2 \times \mathbb{R}^2)$, $\alpha_t = 1$ for all t and for appropriate initial conditions, (4.4a–c) reduces to recursive least squares estimation of θ. This can be seen by checking that $\theta_t = (\sum_{j=1}^t z_{j-1}z_{j-1}')^{-1}(\sum_{j=1}^t z_{j-1}y_j)$ is a solution to (4.4a–c). We refer to (4.4a–c) as the RLS algorithm. Other choices of α_t yield weighted least squares. In the recursive form (4.4) the estimate θ_t is revised in response to the last forecast error ϵ_t.

In considering the limit points of the above real time learning rule, we will limit the discussion to the case of (asymptotically) stationary AR(1) solutions. An extension to the nonstationary case can easily be developed using transformed variables, see Evans and Honkapohja (1993).

Proposition 4.1. Let $(a(\rho), \rho) \equiv \theta_f$ *denote a particular AR*(1) *RE solution to the model* (2.1) *and suppose that* $|\rho| < 1$. *If* θ_f *is weakly E-stable then there exists a nontrivial projection facility such that under* (4.3)–(4.4) *we have* $\theta_t \to \theta_f$ *with probability* 1.

The basic idea of the proof[16] is that, provided certain technical conditions are met, the convergence of (θ_t, R_t) is determined by the stability of Ljung's 'associated differential equation'

$$d\theta/d\tau = R^{-1}M_z(\theta)(T(\theta) - \theta), \tag{4.5a}$$

[16] Proofs of all propositions in this section are in Appendix C.

$$dR/d\tau = M_z(\theta) - R, \tag{4.5b}$$

where $M_z(\theta) = E[z_t(\theta)z_t(\theta)']$ and $\theta' = (a, b)$. Here $z_t(\theta)$ is the process generated by (4.3) and (4.4d) with θ_t replaced by fixed θ.

Essentially (4.5) is obtained by substituting (4.4c) and (4.4d) into (4.4a), replacing $\theta_t - \theta_{t-1}$ by $d\theta/d\tau$ and $R_t - R_{t-1}$ by $dR/d\tau$, fixing (θ_{t-1}, R_{t-1}) at (θ, R), and replacing α_t by 1 and stochastic quantities by the limit of their expectations. It can be shown that local stability of (θ_f, R_f) in (4.5), where $R_f = M_z(\theta_f)$, is governed by stability of θ_f in the differential equation (2.7) of section 2. But this is just our weak E-stability condition.

Proposition 4.1 establishes a local convergence theorem for weakly E-stable stationary AR(1) solutions. Comparison of (4.5) and (2.7) makes transparent the deep connection between E-stability and the ordinary differential equation approach for showing convergence of recursive algorithms. We remark that the size of the projection facility is determined by the basin of attraction of θ_f.

Corresponding instability results for AR(1) learning now follow directly from Proposition 2 of Marcet and Sargent (1989a).[17] The following proposition in essence says that estimates do not converge to nonrational solutions or to E-unstable REE.

Proposition 4.2. Let $\hat{\theta}' = (\hat{a}, \hat{b})$ satisfy the stationarity condition $|T_{b_1}(\hat{b})| < 1$, and let θ_t be generated by (4.3)–(4.4). (i) If $\hat{\theta} \in \text{int}(D_2)$ is not an AR(1) RE solution then $\theta_t \to \hat{\theta}$ with probability 0. (ii) if $\hat{\theta}$ is an E-unstable AR(1) RE solution (and the modulus of the maximal root of $DT(\hat{\theta}) \neq 1$) then $\theta_t \to \hat{\theta}$ with probability 0.

4.2. Higher order AR and ARMA estimation

We now consider two further issues in real time learning dynamics: (i) Are the AR(1) solutions locally stable under real time learning rules in which the agents overparametrize by estimating a higher order AR process or an ARMA process? (ii) Are the ARMA continua of solutions stable under real time learning dynamics? In both cases we argue that the answer is governed by the E-stability conditions derived in earlier sections. For (i) this can be demonstrated by extending earlier arguments. For (ii) Ljung's theorems cannot be applied for technical reasons, and we must rely on simulations. Before taking up either issue we must specify the econometric learning algorithm.

Suppose that agents have a perceived law of motion of the general form (4.1). In considering the method of econometric estimation of the parameters we must make a distinction as to whether or not the shocks v_t are observed.

[17] Their assumption A1 is not required, see Ljung (1977, Theorem 2).

If v_t is observed at t then the estimation of the parameters (a, b, c, d) can be carried out using the RLS algorithm (4.4) with the state vector z_t augmented to include lags of y_t, v_t, and w_t and with θ_t augmented to include the additional parameters. The actual law of motion of the system is given by (4.4), where we replace (4.3) by

$$z_{t-1}' = (1, y_{t-1}, \ldots, y_{t-s}, v_{t-1}, \ldots, v_{t-r}, w_{t-1}, \ldots, w_{t-q}), \tag{4.6a}$$

$$\theta_t' = (a_t, b_{1,t}, \ldots, b_{s,t}, c_{1,t}, \ldots, c_{r,t}, d_{1,t}, \ldots, d_{q,t}), \tag{4.6b}$$

and where $T(\theta)$ is the mapping discussed in section 2 and formally given in Appendix B.

Before presenting the results, some discussion is needed on the permitted overparametrizations in our learning algorithms. A stationary AR(1) process $y_t = \rho y_{t-1} + v_t$ is equivalent to a stationary ARMA$(m+1, m)$ process of the form $c(L)(1-\rho L)y_t = c(L)v_t$ for any degree m lag polynomial $c(L)$.[18] This is in essence an identification problem created by common factors which we wish to avoid here. Therefore, in considering the stability of recursive learning algorithms, we permit only overparametrizations of either the autoregressive or the moving average components, but not both.

Proposition 4.3. Suppose $(a(\rho), \rho)$ *is an AR*(1) *RE solution to the model* (2.1) *with* $|\rho| < 1$. *Suppose that learning dynamics are given by the RLS algorithm, i.e. by* (4.4) *and* (4.6), *where either* $s = 1$ *or there are no moving average terms in* v_t. *If this AR*(1) *solution is strongly E-stable then there exists a nontrivial projection facility such that* $\theta_t \to \theta_f \equiv (a(\rho), \rho, 0, \ldots, 0)'$ *with probability* 1.[19]

If the exogenous shocks are not observed by agents then they must undertake genuine ARMA estimation. Recursive ARMA estimation procedures, which are modifications of the above algorithms, are described in Ljung and Söderström (1983). Applications to learning within rational expectations models are described in Marcet and Sargent (1993). There are two principal algorithms, which Ljung and Söderström refer to as the *pseudo linear regression* (PLR) algorithm and the *recursive prediction error* (RPE) algorithm. We develop formally these algorithms in Appendix C. The basic idea in them is to replace the unobserved v_t with estimates which are the (unfiltered or appropriately filtered) forecast errors ϵ_t.

Remark. Proposition 4.3 holds also for the PLR and RPE algorithms.

Finally we turn to the ARMA(2, 1) and ARMA(3, 2) continua of solutions. Technical complications, due to the continua of solutions and zero eigenvalues, prevent the application of Ljung's (1977) theorems. Thus we revert to

[18] However, for almost all initial conditions these processes are only equivalent asymptotically.
[19] Instability results, along the lines of Proposition 4.2, can also be obtained.

simulations, which suggest that E-stability does appear to govern convergence under real-time learning dynamics.

4.3. Simulation results

Simulations can provide the following information: (1) local stability of econometric learning dynamics in cases where theoretical results could not be obtained, (2) global analysis of learning dynamics, (3) information on the speed of convergence. We illustrate the findings with two examples. Except where indicated, the simulations do not employ a projection facility.

Example 1 (A case of a weakly E-stable ARMA continuum). Model (2.1) was studied with parameters

$$\alpha=0, \quad \delta=-0.48, \quad \beta_0=2, \quad \beta_1=-1.2 \quad \text{and} \quad \beta_2=-4.$$

The exogenous shock v_t was assumed to be independent normal white noise with standard deviation $\sigma=0.1$ and we allow for the presence of an observable sunspot w_t, generated with the same distribution as v_t but independent of the v_t process. This model has a continuum of ARMA(2, 1) solutions of the form

$$y_t=0.5y_{t-1}-0.15y_{t-2}+v_t+\xi v_{t-1}+\kappa w_{t-1},$$

with ξ and κ free. This set of solutions is stationary and forms a weakly E-stable class.[20] We focus on the issue of local stability under econometric learning dynamics of this ARMA(2, 1) class.

Simulations were conducted using the RPE algorithm without projection facility to estimate ARMA(2, 1) or ARMA(3, 1) models, allowing for a sunspot.[21] Whether there is convergence to a member of the ARMA(2, 1) solution class depends on the initial point relative to the domain of attraction (and on the sequence of random shocks). Fig. 1 shows a typical path, over the first 5000 periods, of an estimated ARMA(2, 1) model with the starting point

$$(a(0), b1(0), b2(0), c1(0), d1(0))=(0, 0.4, -0.25, 0.3, 0.5).$$

The path seems to be clearly converging toward the ARMA(2, 1) class of solutions. This is what we expect since the solution class is weakly E-stable. The simulation was extended to 40,000 periods.[22] The $t=40{,}000$ values for $(a, b1, b2, c1, d1)$ of

[20]There is also an AR(1) solution and an ARMA(3, 2) class of solutions.

[21]The initial $R(0)$ was set at the $E\psi\psi'$ corresponding to initial a, b, c parameters. For the definition of ψ see Appendix C. $\alpha(t)$ was set at $\alpha(t)=1$ for all t and the initial t was set at $t=300$.

[22]The very long simulations were done to be sure of the asymptotics.

$$(-0.0001, 0.5021, -0.1494, 0.4590, 0.4953)$$

appear to confirm the stability under recursive ARMA(2, 1) estimation of this set of ARMA(2, 1) solutions. Note that the final values of the coefficients of v_{t-1} and w_{t-1} depend on the initial values of all parameters.

When we overparametrize with ARMA(3, 1) estimation and start from a point near the ARMA(2, 1) equilibrium, a typical simulation over the first 40,000 periods is shown in Fig. 2a–b. The initial parameter values were chosen to be

$$(a(0), b1(0), b2(0), b3(0), c1(0), d1(0)) = (0, 0.5, -0.15, -0.02, 0.3, 0.5).$$

After 40,000 periods the system has diverged from the ARMA(2, 1) equilibria.[23] At $t = 40{,}000$ the values of $(a, b1, b2, b3, c1, c2, d1, d2)$ are

$$(-0.0004, -0.3792, 0.2078, -0.1022, 0.3549, 0.3309).$$

This is again the expected result, given the failure of the solution to be strongly E-stable.[24]

Example 2 (A case with three real stationary AR(1) solutions, two of which are strongly E-stable). Consider (2.1) with parameters $\alpha = 0$, $\delta = 1$, $\beta_0 = -3.53968254$, $\beta_1 = 6.66666667$, $\beta_2 = -3.17460318$. This yields AR(1) solutions with roots

$$\rho_1, \rho_2, \rho_3 = 0.5, 0.7, 0.9.$$

In each REE we have $a(\rho_i) = 0$. The roots 0.5 and 0.9 are strongly E-stable and 0.7 is E-unstable.

First we illustrate the results of section 4 when a projection facility is employed for the AR(1) coefficient b_1. We choose the projection facility $D_2 = [0.73, 0.97]$ and $D_1 = (0.72, 0.98)$. These sets lie inside the basin of attraction of $\rho_3 = 0.9$ and therefore RLS estimates will converge to this solution. Simulations were conducted using the RLS procedure to estimate an AR(1) process with intercept. v_t was generated by normal white noise, with standard deviation 0.5,[25] and the initial value of b_1 set at 0.74 in $t = 20$. Simulations varied substantially in terms of speed of convergence. An example which converges fairly quickly is shown in Fig. 3. In this simulation

[23] In fact, it has clearly moved to a neighborhood of a point in the ARMA(3, 2) class.

[24] The situation with ARMA(3, 1) estimation is in fact even more complex. When a small projection facility is employed, bounding $b2$ between -0.2 and -0.1, simulations showing apparent convergence to the ARMA(2, 1) solution have been obtained. We interpret this as due to the one-sided stability/instability of the solution with respect to strong E-stability. In the convergent cases the projection facility keeps the system near the equilibrium and the motion is eventually caught by a stable trajectory.

[25] The initial $R(0)$ was set at the identity matrix and $\alpha(t) = 1$ for all t.

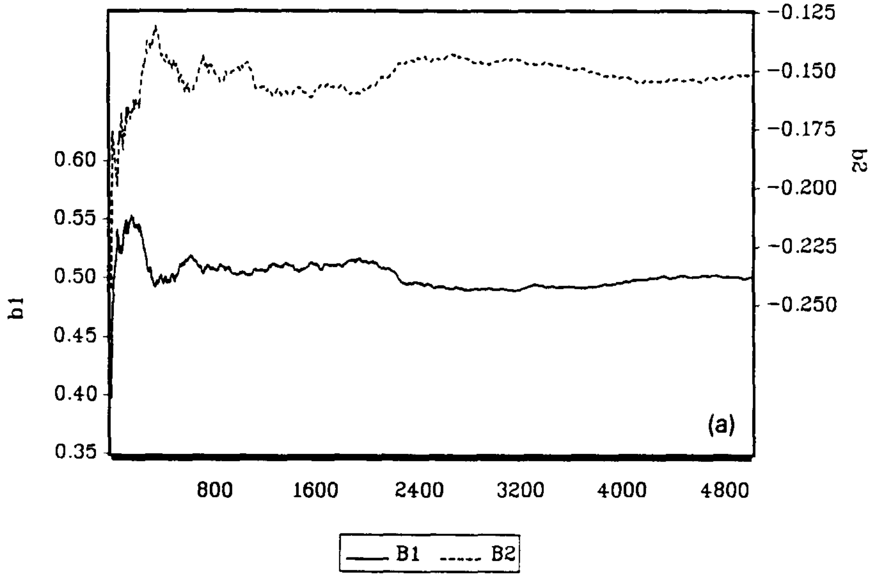

Fig. 1a. ARMA(2, 1) with sunspot. 5000 periods.

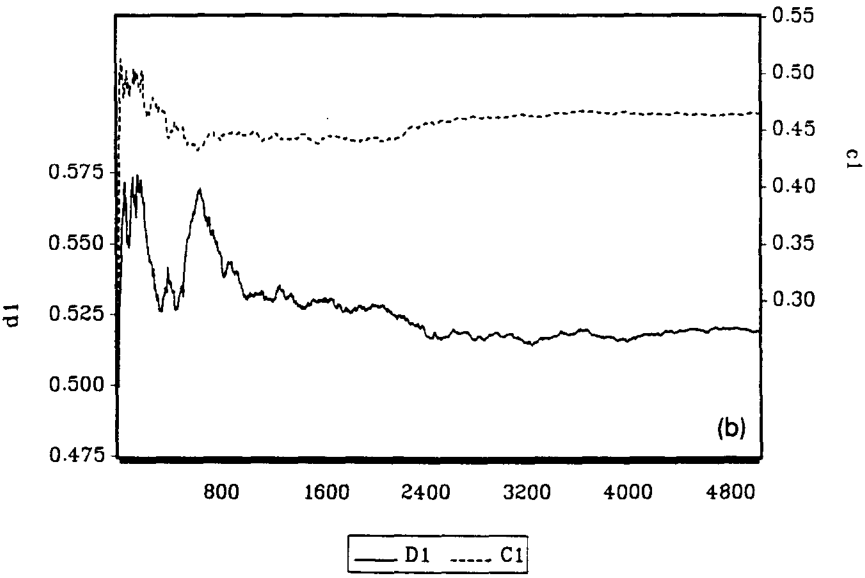

Fig. 1b. ARMA(2, 1) with sunspot. 5000 periods.

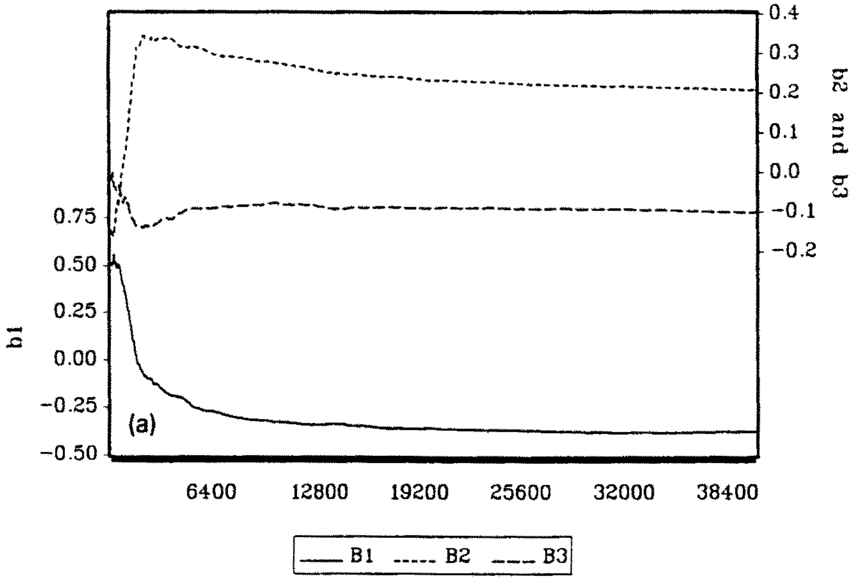

Fig. 2a. ARMA(3, 1) with sunspot. 40,000 periods.

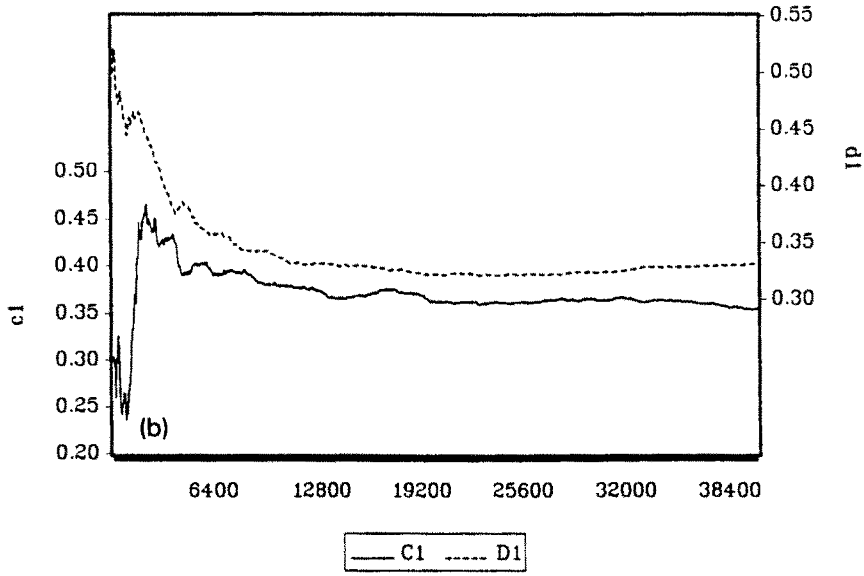

Fig. 2b. ARMA(3, 1) with sunspot. 40,000 periods.

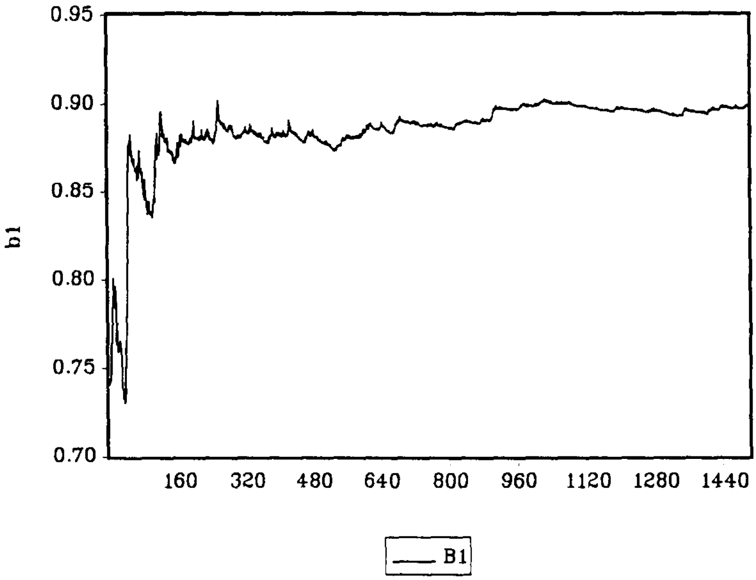

Fig. 3. AR(1) with projection facilities. 1500 periods.

the projection facility is employed only once, after 36 periods, when $b1$ would otherwise have fallen below 0.72. Thereafter convergence toward $b_1=0.9$ is fairly smooth.

Using simulations we can also consider global learning dynamics for this example. Initial parameter estimates are now set at the unstable root $(a, b_1)=(0, 0.7)$ at $t=1$ and no projection facility is employed. Agents are again assumed to estimate an AR(1) process using RLS.

Table 1 gives the distribution of the autoregressive coefficient $b_1(t)$, based on 1000 simulations. After 50 periods the distribution shows little pattern over (0.25, 0.99). After 1000 periods, the importance of E-stability becomes clear, with 2 peaks around the E-stable points and a trough in the distribution near the E-unstable starting point. The bimodal shape is intensified at 10,000 periods. The speed of convergence is slow, but of course the length of the 'period' could be quite short in applications.

5. An economic application

In this section we provide a brief application of the preceding results to an open-economy macroeconomic model illustrating the relevance of the model (2.1). The model is a variant of the Dornbusch (1976) overshooting model:

$$p_t - p_{t-1} = \pi(\mathrm{E}_{t-1} d_t) + u_t, \tag{5.1}$$

$$d_t = -\gamma(r_t - \mathrm{E}_t p_{t+1} + p_t) + \eta(e_t - p_t), \tag{5.2}$$

Table 1
Distribution of $b1$ for AR(1) estimation.[a]

Value of $b1$	Distribution of $b1$, in percent after 50, 1000 and 10,000 periods		
	50 periods	1000 periods	10,000 periods
<0.35	11.6	0.1	0.0
0.35–0.45	15.1	11.6	1.0
0.45–0.55	17.3	30.0	41.9
0.55–0.65	14.3	12.8	11.9
0.65–0.75	13.9	11.0	8.3
0.75–0.85	14.0	11.7	9.6
0.85–0.95	12.2	21.9	27.3
>0.95	1.6	0.9	0.0

[a] Based on 1000 simulations with model parameters as in text. E-stable roots at 0.5 and 0.9. E-unstable root at 0.7. Initial $b1$ set at 0.7.

$$r_t = \lambda^{-1}(p_t - \vartheta p_{t-1}), \tag{5.3}$$

$$r_t = \mathrm{E}_t e_{t+1} - e_t. \tag{5.4}$$

All variables, except interest rates, are taken as logarithms, and the model is described in deviation form. Eq. (5.1) is a price adjustment rule of the usual kind, according to which prices respond to expected excess demand. Output is taken to be constant. Eq. (5.2) gives the aggregate demand function, dependent on real interest rates and international competitiveness. Eq. (5.3) gives the LM equation in which we assume that money supply accommodates past inflation to the extent described by parameter ϑ. Finally, (5.4) is the usual interest rate parity or arbitrage condition.

In the basic formulation all the parameters π, γ, η, λ, ϑ are taken to be positive, but for later purposes we note that in somewhat exceptional circumstances the competitiveness parameter η might be negative. This could come about when the economy is strongly dependent on an imported raw material and the demand elasticity for its exports is low. A detailed discussion of this argument is provided in Bruno and Sachs (1985, pp. 104–105).

Elementary but tedious calculations lead to the price equation:

$$\begin{aligned} p_t = {} & (1 + \pi\gamma\vartheta\lambda^{-1} + \pi\vartheta\eta\lambda^{-1})p_{t-1} \\ & -(1 + \pi\gamma\vartheta\lambda^{-1} + \pi\eta\lambda^{-1} + \pi\gamma\lambda^{-1} + \pi\gamma + \pi\eta)\mathrm{E}_{t-1}p_t \\ & +(1 + \pi\eta + 2\pi\gamma + \pi\gamma\lambda^{-1})\mathrm{E}_{t-1}p_{t+1} - \pi\gamma E_{t-1}p_{t+2} + u_t. \end{aligned} \tag{5.5}$$

This is of type (2.1). In the illustrations below we have picked values for π, γ, and λ which might be considered as fairly reasonable. We then vary policy

Table 2

Strong E-stability and stationarity of AR(1) equilibria.[a]

Parameters	AR(1) roots	E-stability	Stationarity
$\vartheta=0.5$	1.250	no	no
$\eta=0.2$	1.043	no	no
	0.384	yes	yes
$\vartheta=0.5$	1.067	yes	no
$\eta=-0.1$	0.898	no	yes
	0.512	no	yes
$\vartheta=1.1$	0.989	yes	yes
$\eta=-0.1$	0.772	no	yes
	0.716	yes	yes

[a] Maintained parameter values: $\pi=1.5$, $\gamma=1.5$, $\lambda=10$.

responsiveness ϑ and competitiveness η to show how various possibilities of E-stable equilibria can arise. The results for the AR(1) equilibria are displayed in Table 2.

On the first row there is a unique, strongly E-stable equilibrium and it is stationary. The other rows show that for $\eta<0$ and/or $\vartheta>1$ non-stationary strongly E-stable or multiple strongly E-stable cases can arise. It is also possible to get weakly E-stable ARMA(2, 1) solutions. For example, when $\pi=1.5$, $\gamma=1.5$, $\lambda=10$, $\vartheta=1.1$ and $\eta=-0.1$, the stationary ARMA(2, 1) class

$$p_t=1.7618p_{t-1}-0.7642p_{t-2}+v_t+\xi v_{t-1}+\kappa w_{t-1},$$

with ξ and κ free, is weakly E-stable. Thus for some choices of parameters it is possible for econometric learning procedures to converge to sunspot solutions, although this result is not robust to overparametrization of the learning algorithm.

Although somewhat extreme parameter values are used to obtain the more exotic phenomena, we believe this example illustrates the range of possibilities that may be encountered in applied models.

6. Conclusions

We have explored the conditions for convergence of real-time learning algorithms in dynamic linear models when multiple REE prevail. The results indicate that convergence of econometric learning dynamics is governed by E-stability conditions. The theoretical basis for this relationship is clear for the finite set of AR(1) equilibria. For technical reasons the issue remains open in theory for continua of REE, including sunspot solutions, though simulations support the connection.

It must be emphasized that stability can depend on how the agents parametrize expectations and, in particular, on possible overparametrization of a given solution. The applied macroeconomist will typically focus on minimal state variable solutions, i.e. the AR(1) solutions for the model considered here, and the stability of the AR(1) solutions can be robust to overparametrization. In many cases this principle of selecting stable solutions provides a unique choice. However, as we have seen, this need not be so if the reduced form places a large enough weight on expectational variables. If multiple strongly stable REE are present, this feature is inherent to the model and cannot be avoided by the practitioner.

For some practical questions the continua of solutions may be of special interest. For example, it may be desirable to take seriously the sunspot solutions. If so, the analysis of learning is still relevant. Stability under learning again narrows the set of attainable solutions, and in some cases learning can converge to these solutions. However, the issue of robustness to overparametrization is critical here, as we have shown that the solutions continua can be weakly but are not strongly E-stable.

We anticipate that the full range of features discovered in this paper will typically be present in more complex linear models, even perhaps including large-scale empirical models. The generalization of our findings clearly warrants further study. However, the tools presented in this paper can already be applied to particular models of interest.

Appendix A: Derivation of the sunspot equilibria (2.2)

Following Evans and Honkapohja (1986) denote by y_t^F the 'longest' ARMA solution generated by the disturbance term v_t of model (2.1). This has the form (2.2) without the sunspot terms $d_1 w_{t-1} + d_2 w_{t-2}$. Let y_t^S denote any sunspot term (which is a process not linearly dependent on v_t's as such a dependence would be included in the ARMA solution just mentioned). If y_t^S satisfies the homogeneous equation corresponding to (2.1) then, by linearity, a general solution to (2.1) is given by $y_t^F + y_t^S$. Write y_t^S as an arbitrary linear function of arbitrary innovations w_t and past values of y_t^S:

$$y_t^S = \sum_{j=1}^{r} \gamma_j y_{t-j}^S + \sum_{s=0}^{T} \pi_s w_{t-s}. \tag{A.1}$$

From (A.1) one computes $\mathrm{E}_{t-1} y_t^S = y_t^S - \pi_0 w_t$, $E_{t-1} y_{t+1}^S = y_{t+1}^S - (\pi_1 + \gamma_1 \pi_0) w_t - \pi_0 w_{t+1}$, etc. Substituting into the homogeneous equation yields

$$\begin{aligned} y_t^S = & -\beta_2^{-1}\beta_1 y_{t-1}^S + \beta_2^{-1}(1-\beta_0) y_{t-2}^S - \beta_2^{-1}\delta y_{t-3}^S + \pi_0 w_t \\ & + [\pi_1 + (\gamma_1 + \beta_2^{-1}\beta_1)\pi_0] w_{t-1} \\ & + [(\gamma_2 + \gamma_1^2)\pi_0 + \gamma_1 \pi_1 + \beta_2^{-1}\beta_1(\pi_1 + \gamma_1 \pi_0) + \beta_2^{-1}\beta_0 \pi_0] w_{t-2}. \end{aligned} \tag{A.2}$$

Solutions to the homogeneous equation must also satisfy $y_t^2 - \mathrm{E}_{t-1} y_t^S = 0$, so that $\pi_0=0$, while π_1 and π_2 can be freely chosen. Therefore, the terms involving w_{t-i}, for $i=0,1,2$, take the form $d_1 w_{t-1}+d_2 w_{t-2}$, with d_1 and d_2 arbitrary, and we have (2.2).

Appendix B: E-stability results

In the general case the mapping $\theta^*=T(\theta)$ takes the form

$$a^*=\alpha+\beta_0 a+\beta_1(b_1 a+a)+\beta_2[b_1(b_1 a+a)+b_2 a+a],$$

$$b_1^*=\delta+\beta_0 b_1+\beta_1(b_1 b_1+b_2)+\beta_2[b_1(b_1 b_1+b_2)+b_2 b_1+b_3],$$

$$b_i^*=\beta_0 b_i+\beta_1(b_1 b_i+b_{i+1})+\beta_2[b_1(b_1 b_i+b_{i+1})+b_2 b_i+b_{i+2}]$$

$$\text{for} \quad i=2,\dots,s-2,$$

$$b_{s-1}^*=\beta_0 b_{s-1}+\beta_1(b_1 b_{s-1}+b_s)+\beta_2[b_1(b_1 b_{s-1}+b_s)+b_2 b_{s-1}],$$

$$b_s^*=\beta_0 b_s+\beta_1(b_1 b_s)+\beta_2[b_1(b_1 b_s)+b_2 b_s],$$

$$c_i^*=\beta_0 c_i+\beta_1(b_1 c_i+c_{i+1})+\beta_2[b_1(b_1 c_i+c_{i+1})+b_2 c_i+c_{i+2}]$$

$$\text{for} \quad i=1,\dots,r-2,$$

$$c_{r-1}^*=\beta_0 c_{r-1}+\beta_1(b_1 c_{r-1}+c_r)+\beta_2[b_1(b_1 c_{r-1}+c_r)+b_2 c_{r-1}],$$

$$c_r^*=\beta_0 c_r+\beta_1(b_1 c_r)+\beta_2[b_1(b_1 c_r)+b_2 c_r],$$

$$d_i^*=\beta_0 d_i+\beta_1(b_1 d_i+d_{i+1})+\beta_2[b_1(b_1 d_i+d_{i+1})+b_2 d_i+d_{i+2}]$$

$$\text{for} \quad i=1,\dots,q-2,$$

$$d_{q-1}^*=\beta_0 d_{q-1}+\beta_1(b_1 d_{q-1}+d_q)+\beta_2[b_1(b_1 d_{q-1}+d_q)+b_2 d_{q-1}],$$

$$d_q^*=\beta_0 d_q+\beta_1(b_1 d_q)+\beta_2[b_1(b_1 d_q)+b_2 d_q]. \tag{B.1}$$

The differential equation (2.7), with T given by (B.1), has the property that the equations for b are independent of a, c and d. Close inspection of the structure also reveals that, for assessing stability, we can assume $s=4$, $r=q=3$, without loss of generality.

The first step is to linearize (2.7). Since the subsystem for variables $b=(b_1,\dots,b_4)'$ is independent of the other variables we write it in vector form as $\dot{b}=f(b)$. Denoting any equilibrium point by $\bar{b}$ and the deviation variables by $b^\dagger=b-\bar{b}$ the linearization takes the form $\dot{b}^\dagger=(\partial f(\bar{b})/\partial b)b^\dagger$, where the matrix $\partial f(\bar{b})/\partial b$ of the partial derivatives $\partial f_i/\partial b_j$ is computed to be

$$\begin{bmatrix} \beta_0+2\beta_1\bar{b}_1+\beta_2(3\bar{b}_1^2+2\bar{b}_2)-1 & \beta_1+2\beta_2\bar{b}_1 & \beta_2 & 0 \\ \beta_1\bar{b}_2+2\beta_2\bar{b}_2\bar{b}_1+\beta_2\bar{b}_3 & \beta_0+\beta_1\bar{b}_1+\beta_2(\bar{b}_1^2+2\bar{b}_2)-1 & \beta_1+\beta_2\bar{b}_2 & \beta_2 \\ \beta_1\bar{b}_3+2\beta_2\bar{b}_3\bar{b}_1 & \beta_2\bar{b}_3 & \beta_0+\beta_1\bar{b}_1+\beta_2(\bar{b}_1^2+\bar{b}_2)-1 & \beta_1+\beta_2\bar{b}_1 \\ 0 & 0 & 0 & \beta_0+\beta_1\bar{b}_1+\beta_2(\bar{b}_1^2+\bar{b}_2)-1 \end{bmatrix} \tag{B.2}$$

Writing $a^{\dagger}=a-\bar{a}$ we obtain the linearized equation

$$\begin{aligned} \dot{a}^{\dagger}&=[\beta_0+\beta_1(1+\bar{b}_1)+\beta_2(1+\bar{b}_1(1+\bar{b}_1)+\bar{b}_2)-1]a^{\dagger} \\ &\quad+[\beta_1\bar{a}+\beta_2\bar{a}(1+2\bar{b}_1)]b_1^{\dagger}+\beta_2\bar{a}b_2^{\dagger}\equiv \mathbb{A}a^{\dagger}+\mathbb{B}b^{\dagger}. \end{aligned} \tag{B.3}$$

We omit explicit linearization of $\dot{d}=g(d,b)$ and $\dot{c}=g(c,b)$.

Proof of Proposition 3.1. To assess strong E-stability we set $\bar{b}_1=\rho_1$, $\bar{b}_i=0$ for $i=2,3,4$, $\bar{c}=0$, and $\bar{d}=0$. Then

$$\begin{aligned} \det[\partial f(\bar{b})/\partial b-\lambda I]&=(\beta_0+\beta_1\bar{b}_1+\beta_2\bar{b}_1^2-1-\lambda)^3 \\ &\quad\times(\beta_0+2\beta_1\bar{b}_1+3\beta_2\bar{b}_1^2-1-\lambda). \end{aligned}$$

Using the relationships (2.3) between β_i's and ρ_i's from section 2, we have 3 roots equal to $\beta_0+\beta_1\rho_1+\beta_2\rho_1^2-1=\beta_2\rho_2\rho_3$ and one root equal to $\beta_0+\beta_1\rho_1+3\beta_2\rho_1^2-1=\beta_2[-\rho_1\rho_2-\rho_1\rho_3+\rho_2\rho_3+\rho_1^2]$. Since these expressions are necessarily real, their negativity is a necessary condition for strong E-stability. For weak E-stability we set $s=1$ so that only the latter single root appears. (In what follows we only consider the generic case that these roots are non-zero.)

Finally we consider the other variables. From eq. (B.3) for $a^{\dagger}$ we have an additional stability condition $\mathbb{A}<0$ or, writing it in terms of ρ_i's and simplifying, $\beta_2[\rho_2\rho_3-\rho_2-\rho_3+1]<0$. The equations for vectors c and d (omitted) provide no further conditions.

Proof of Corollary 2 (sketch). Consider the case $\beta_0=\beta_1=0$ (with $\beta_2\neq 0$). An AR(1) root ρ satisfies $\beta_2\rho^3-\rho+\delta=0$. If $\beta_2>0$ is sufficiently small there are 3 real roots which satisfy $\rho_1<-1<\rho_2<1<\rho_3$. If $\beta_2<0$ there is a unique real root $-1<\rho_2<1$ and the complex roots have squared modulus β_2^{-1}. It is readily verified that the ρ_2-AR(1) solution is uniquely strongly E-stable, and the Corollary follows by continuity of roots and eigenvalues.

Proof of Proposition 3.2. For E-stability of the ARMA(3, 2) solution class we first evaluate the coefficient matrix for b at the ARMA(3, 2) equilibrium values yielding

$$\begin{bmatrix} 3(\beta_0-1)+\beta_1^2\beta_2^{-1} & -\beta_1 & \beta_2 & 0 \\ \beta_1\beta_2^{-1}(1-\beta_0)-\delta & 3(\beta_0-1) & \beta_2 & \beta_2 \\ \beta_1\beta_2^{-1}\delta & -\delta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} & \vdots & 0 \\ A & \vdots & \beta_2 \\ & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix}. \tag{B.4}$$

Eq. (B.4) has one zero eigenvalue, but this does not arise with weak E-stability (when $s=3$ and $b_4 \equiv 0$ is imposed). Then only the first three equations having the coefficient matrix A are relevant.

The parameter domain for weak E-stability of the ARMA(3, 2) solution class is given by:

(a) A is a stable matrix and

(b) $\beta_2 < 0$.

Condition (a) is immediate from the above analysis of the b subsystem. Condition (b) then follows from the a equation (B.3).

Moving to consider the systems for c and d, we note that with $\bar{b}_1 = -\beta_1\beta_2^{-1}$, $\bar{b}_2 = (1-\beta_0)\beta_2^{-1}$, $\bar{b}_3 = -\delta\beta_2^{-1}$, $\bar{b}_4 = 0$, the linearization is trivial, and one must revert to the original equations. The equation for c_2 is a linear differential equation with time-varying coefficients via the independently evolving variables b_1 and b_2, so it can be directly integrated. Using an argument analogous to that in Evans and Honkapohja (1992a, pp. 5–6) it is seen that c_2 converges. Given the time paths $c_2(\tau)$ the equation for c_1 is a linear equation with variable coefficients with time-dependence through b_1, b_2 and c_2, so it can also be directly integrated and $c_1(\tau)$ will converge to a finite value. Thus, c and d converge when the subsystem for b is stable.

Proof of Proposition 3.3. For the ARMA(2, 1) solutions, first evaluate the coefficient matrix (B.2) at the appropriate REE values of the AR parameters $\bar{b}$. The resulting matrix can be written

$$\begin{bmatrix} B & \vdots & \beta_2 & 0 \\ & \vdots & \beta_1+\beta_2\bar{b}_1+\beta_2 & \beta_2 \\ \cdots & \cdots & \cdots & \cdots \\ & \vdots & 0 & \beta_1+\beta_2\bar{b}_1 \\ 0 & \vdots & 0 & 0 \end{bmatrix}, \tag{B.5}$$

where B is the 2×2 matrix in the top left corner. (B.5) has a zero eigenvalue of multiplicity two, which complicates only the analysis of strong E-stability of the ARMA(2, 1) solutions. The parameter domain for weak E-stability is given by:

(a) B is a stable matrix, and

(b) $\beta_0+\beta_1(1+\bar{b}_1)+\beta_2(1+\bar{b}_1(1+\bar{b}_1)+\bar{b}_2)-1<0$.

Condition (a) follows from (B.5) and (b) from (B.3). The rest of the proof is analogous to the last stage outlined in the proof of Proposition 3.2 above.

Proof for Proposition 3.4 (remark). It is sufficient to show instability of the subsystem for vector b. Linearization cannot be used as (B.4) has a zero eigenvalue and (B.5) a zero eigenvalue of multiplicity two. The method of proof is the same as in Evans and Honkapohja (1992a). Details are available from the authors.

Appendix C: Real time learning results

Proof of Proposition 4.1. The result is based on Theorem 4 of Ljung (1977) and is most easily demonstrated using the proofs of Propositions 1 and 3 of Marcet and Sargent (1989).[26] The associated differential equation is given by (4.5). For a sufficiently small open interval $I_s(\rho)$ around ρ, (4.5) has a unique equilibrium (θ_f, R_f) on $D_R=(\mathbb{R}\times I_s(\rho))\times(\mathbb{R}^2\times\mathbb{R}^2)$ and $M_z(\theta)$ is nonsingular with $|T_{b_1}(b)|<1$ for $(\theta, R)\in D_R$. By the assumption of weak E-stability (2.7) is locally stable at θ_f. Using the argument of Marcet and Sargent (1989a), Proposition 3, it follows that (4.5) is locally stable at (θ_f, R_f).

Let D_A be the intersection of the domain of attraction of (θ_f, R_f) with D_R. D_A is an open set. It is readily verified that (A1)–(A5) of Marcet and Sargent (1989a) are satisfied on D_A. Hence, as they argue, conditions B of Ljung (1977) are met. To apply Proposition 1 of Marcet and Sargent (1989a) we require also that their assumption A.7.2 is met, i.e. defining D_2 closed, D_1 open and bounded, with $D_2\subset D_1\subset D_A$, such that trajectories of (4.5) starting in D_2 never leave a closed subset of D_1. But the existence of suitable nontrivial D_1 and D_2 follows from local stability of (θ_f, R_f) under (4.5).

We can thus invoke Proposition 1 of Marcet and Sargent (1989a) and conclude that $\theta_t=(a_t, b_t)'$ converges almost surely to $(a(\rho), \rho)'$ as $t\to\infty$. (The statement of Proposition 1 of Marcet and Sargent appears to require that it also be shown that their assumption (A.6) is satisfied. However, it is clear from the proof of their Proposition 1, and from the statement and discussion of Theorem 4 of Ljung, that no additional condition is required).

[26] Marcet and Sargent (1989a) incorporate a 1-period observation lag into their (4a–b). Our (z_t, v_t) corresponds to their $(z_{2,t-1}, u_{t-1})$ and we use θ_t for their β_t. Finally, for their algorithm to correspond exactly to RLS, α_t/t in their R_t equation of (4a) should be $\alpha_t/(t+1)$. This makes no difference asymptotically.

Proof of Proposition 4.3. The RLS algorithm is given by (4.4) and (4.6) with $n=s+r+q+1$ in the projection facility. The associated differential equation continues to be given by (4.5), where $M_z(\theta)=Ez_t(\theta)z_t(\theta)'$ and where now $z_t(\theta)$ is defined as the process generated by (4.6) and (4.4d) with θ_t replaced by a fixed θ. The argument follows the proof of Proposition 4.1 except that $T(\theta)$ is given by (B.1), so that strong E-stability is needed for the stability of (θ_f, R_f) under (4.5).

In the ARMA estimation with unobservable v_t we assume that agents estimate v_t by ϵ_t. The perceived law of motion is then given by (4.1) with ϵ_t in place of v_t. With learning dynamics following the PLR algorithm the actual law of motion is given by (4.4) and

$$z_{t-1}'=(1,y_{t-1},\dots y_{t-s},\epsilon_{t-1},\dots,\epsilon_{t-r},w_{t-1},\dots w_{t-q}), \tag{C.1a}$$

$$\theta_t'=(a_t,b_{1,t},\dots,b_{s,t},c_{1,t},\dots c_{r,t},d_{1,t},\dots,d_{q,t}), \tag{C.1b}$$

with $n=s+r+q+1$ in the projection facility. Initial values for $(\epsilon_0,\dots,\epsilon_{-(r-1)})$ and $(w_0,\dots,w_{-(q-1)})$ are required, and in the simulations they are set at 0. The PLR algorithm differs from RLS only in that the unobserved v_t are replaced by running estimates ϵ_t. In the RPE algorithm (4.4a, b) is replaced by

$$\psi_t=-c_{1,t-1}\psi_{t-1}-\dots-c_{r,t-1}\psi_{t-r}+z_t, \tag{C.2a}$$

$$\bar{\theta}_t=\theta_{t-1}+(\alpha_t/t)(\bar{R}_t)^{-1}\psi_{t-1}\epsilon_t, \tag{C.2b}$$

$$\bar{R}_t=R_{t-1}+(\alpha_t/t)(\psi_{t-1}\psi_{t-1}'-R_{t-1}/\alpha_t). \tag{C.2c}$$

Learning dynamics under the RPE algorithm are thus specified by (4.4c–e), (C.1) and (C.2). The RPE algorithm augments the PLR algorithm with an extra state variable, ψ_t, defined as z_t filtered by the inverse of the estimated moving average polynomial. ψ_t then replaces z_t in the recursive formulae for (θ_t, R_t). The advantage of the RPE algorithm is that it is an approximation to maximum likelihood estimation (when used to estimate an exogenous ARMA process), see Ljung and Söderström (1983, esp. ch. 3.7) and Marcet and Sargent (1993). The proof of the remark at the end of section 4.2 is available on request.

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