Estimation, Solution and Analysis of Equilibrium Monetary Models

Assignment 3: Solution and Analysis of a Simple Dynamic General Equilibrium Model: a Tutorial

The primary purpose of this tutorial is to provide the reader with hands-on experience in solving a dynamic, general equilibrium model by log-linearization. To make the discussion as accessible as possible, the analysis is done in a very simple dynamic general equilibrium model. Despite this, the tools that are presented are easily generalized so that they work with the most elaborate general equilibrium models currently in use: models with employment and unemployment, with a banking sector, financial frictions, multiple goods-producing sectors, etc.[1] It is hoped that the student who works through this tutorial will be in a position to quickly jump to these more complicated applications if he/she so chooses. Of course, solving more complicated models involves additional technicalities not addressed here. However, the available software is advancing so rapidly that soon the technical expertise required to analyze even very complicated models will not be much greater than what is covered in this tutorial.[2]

Although the model economy used below is quite simple, it is nevertheless rich enough to convey some key forces that lie at the heart of modern general equilibrium models. You will see how the operation of rate of return effects and wealth effects shape the response of the economy to a technology shock.

The tutorial below focuses on two versions of a simple growth model. Questions 1 - 10 analyze a version in which the shock is stationary. However, in the analysis of time series data, it is often useful to incorporate non-stationary shocks. Accordingly, the analysis is redone in Question 11 in a version of the model where the shock has a unit root. This presents an interesting technical problem. The methods discussed throughout this tutorial are based on linearizing the model economy about a steady state. When a shock has a unit root, then the economy does not, strictly speaking, have a steady state. In Question 11, we see that if variables are scaled appropriately, then there may yet be a sense in which the model economy has a steady state after all, even in the presence of a unit root.

The 11 questions in the tutorial involve deriving formulas and entering them

[1]A direct generalization of the solution method discussed here is presented in Christiano (2000). The method is similar to others. See, for example the articles in the October 2000 issue of *Computational Economics*. For an application of the method to a model with production, banking, investment, financial frictions and money, see Christiano, Motto and Rostagno (2004).

[2]An example of advances in software is the suite of programs in DYNARE. To find out more about this, go to the following web page: http://www.cepremap.cnrs.fr/~michel/dynare/

into MATLAB programs. To minimize time spent on MATLAB technicalities, most of the MATLAB programs have been written and you are asked to just program in the formulas in specific places.

1. Description of the Model

Consider an economy in which households seek to maximize utility of consumption, C_t :

$$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t),\ \frac{C^{1-\sigma}}{1-\sigma}. \tag{1.1}$$

There is a resource constraint, which says that consumption plus investment, I_t, must not exceed total output of goods, Y_t :

$$C_t + I_t \leq Y_t. \tag{1.2}$$

Goods are produced using physical capital, k_t. In addition, there is a technology shock, ε_t :

$$Y_t = \varepsilon_t \left(u_t k_t\right)^{\alpha}. \tag{1.3}$$

Here, u_t denotes the rate at which capital is utilized. Investment is composed of two parts:

$$I_t = I_t^k + I_t^u \tag{1.4}$$

The component, I_t^k, is used to increase the quantity of physical capital:

$$k_{t+1} = (1-\delta)k_t + I_t^k, \tag{1.5}$$

where $\delta \in (0,1)$ is the rate of depreciation on physical capital. The component of investment, I_t^u, reflects maintenance expenditures that arise as capital is utilized more intensely:

$$I_t^u = a\left(u_t\right) k_t. \tag{1.6}$$

Here, the function, a, is increasing, $a' > 0$, and convex, $a'' > 0$. It is convenient to restrict the functional form of a so that in a steady state, $u_t = 1$. The following functional form is useful:

$$a(u) = 0.5b\sigma_a u^2 + b(1-\sigma_a)u + b((\sigma_a/2) - 1), \tag{1.7}$$

where σ_a is a parameter that controls the curvature of a at steady state, $u_t = 1$ (curvature is defined as $a''(u)u/a'(u)$). Also, b, is a function of the other parameters of the model, and it must be chosen so that $u = 1$ in a steady state.

To complete the description of the model, we have to say something about the stochastic process governing ε_t. Let ε denote the unconditional mean of ε_t, so that $\varepsilon = E\varepsilon_t$. Define $\hat{\varepsilon}_t = (\varepsilon_t - \varepsilon)/\varepsilon$. Then, we suppose:

$$\hat{\varepsilon}_t = \rho\hat{\varepsilon}_{t-1} + e_t,$$

where e_t is a white noise process. Note that $E\hat{\varepsilon}_t = 0$, which is consistent with our assumption, $\varepsilon = E\varepsilon_t$.

A baseline set of parameter values for the model is, $\sigma = 1$, $\delta = 0.02$, $\alpha = 0.36$, $\rho = 0.95$, $\varepsilon = 1$, $\sigma_a = 0.1$, $\beta = 1.03^{-0.25}$. These parameters correspond to a quarterly time period for the model.

2. Questions and Analysis

1. Show that $a(1) = 0$, $a'(1) = b$, $a''(1)/a'(1) = \sigma_a$.

2. **First order conditions**. Derive the first order necessary conditions for optimization. Do this by first using the constraints to express C_t as a function of k_t, k_{t+1}, u_t, ε_t. Then, use this to substitute out for C_t in the utility function. Then, differentiate with respect to u_t and k_{t+1} to obtain the following first order conditions:

$$\begin{aligned} v_u(k_t, u_t, \varepsilon_t) &= 0 \\ v_k(k_t, k_{t+1}, k_{t+2}, u_t, u_{t+1}, \varepsilon_t, \varepsilon_{t+1}) &= 0. \end{aligned} \tag{2.1}$$

Display expressions for v_u and v_k.

3. **Analysis of capital utilization**. Multiply the v_u equation by $k^{-\alpha}$. Note that $v_u(k_t, u_t, \varepsilon_t)k_t^{-\alpha}$ has a term with a positive sign (this is the 'marginal benefit of utilization') and a term with a negative sign (the 'marginal cost'). Show that as $u_t \to 0$ the marginal benefit of capital utilization goes to infinity, as $u_t \to \infty$ the marginal benefit goes to zero, and marginal benefits are strictly decreasing for all $u > 0$. Note that the marginal cost is strictly increasing, finite for $u = 0$ and positive for u sufficiently large. From this, you can conclude that for each k_t, ε_t there is a unique u_t where marginal benefit and marginal cost intersect. Show that as ε_t increases the optimal choice of u_t increases. As k_t increases, the optimal choice of u_t *decreases.* Can you provide intuition for this result?

4. **Rate of Return on Capital**. The rate of return on capital, R_t^k, is the marginal product of capital, plus what is left over next period after depreciation and maintenance expenses:

$$R_t^k \equiv MP_{k,t} + 1 - \delta - a(u_t).$$

Here, $MP_{k,t}$ is the marginal product of capital. Given our specification of technology, this is $\alpha\varepsilon_t u_t^\alpha k_t^{\alpha-1}$. If you look carefully at the function, v_k, that you derived above, you will see that it has the following form:

$$u'(C_t) = E_t\beta u'(C_{t+1})R_{t+1}^k.$$

On the left of the above equality, you have the cost of acquiring one extra unit of capital, in utility units. The term on the right describes the net benefits - also in utility units - enjoyed in the next period by acquiring a unit of capital in the current period. There is an expectation operator there, because next period's technology, ε_{t+1}, is uncertain. It is easy to see that the rate of return on capital is an important variable determining how much capital investment there will be. If the rate of return were anticipated to be higher, then consumption would be rearranged so that current consumption is lower (thus, raising the term on the left of the equality) and future consumption is higher (reducing $u'(C_{t+1})$ on the right). This intertemporal reallocation of consumption corresponds to a current increase in investment. So, a higher rate of return on capital leads to an increase in investment by increasing the incentive to invest.

Show that a higher value of u_t leads to a *fall* in the rate of return on investment, $MP_{k,t}+1-\delta-a(u_t)$. (Hint: note that u_t enters this in two places, through $MP_{k,t}$ and $a(u_t)$. Show that $a'(u_t) = MP_{k.t}/u_t$. Show that the derivative of $MP_{k,t}$ with respect to u_t is smaller than $MP_{k,t}$, in steady state.) Conclude that if something caused utilization to rise, the rise in utilization per se would reduce the incentive to invest.

5. **Steady State**. Obtain formulas for the steady state values of C_t, k_t by solving

$$\begin{aligned} v_u(k, u, \varepsilon) &= 0 \\ v_k(k, k, k, u, u, \varepsilon, \varepsilon) &= 0, \end{aligned}$$

where a variable without a time subscript is its value in steady state: a situation in which variables take on a constant value, shocks are replaced by their mean, and the equilibrium conditions are satisfied. How must b be related to the model parameters to ensure that, in steady state, $u_t = 1$? Type your formulas in on lines 19 and 20 of hmk1.m.

We now take a brief break to discuss log-linearization of equations. Here is a very simple example, by way of motivation. Suppose y_t is an endogenous variable, and x_t is exogenous. Economic reasoning leads to the restriction, $f(y_t, x_t) = 0$. Suppose we want to understand how y_t varies with changes in x_t. But, suppose f is quite complicated, so that solving y_t for each new x_t is difficult. A simple procedure is available, if we know what value y_t takes on for some particular x_t, say x. That is, we know $f(y, x) = 0$. We can use this information to approximate how y_t varies with x_t. In particular, replace f with F, the first order Taylor series expansion of f about $y_t = y$, $x_t = x$:

$$\begin{aligned} F(y_t, x_t) &= f(y, x) + f_1(y, x)(y_t - y) + f_2(y, x)(x_t - x) \\ &= f_1(y, x)(y_t - y) + f_2(y, x)(x_t - x), \end{aligned}$$

taking into account that $f(x, y) = 0$. In practice, the variables in F are expressed as a ratio to the values about which the Taylor series expansion is taken:

$$\begin{aligned} F(y_t, x_t) &= f_1(y, x)y(\frac{y_t - y}{y}) + f_2(y, x)x(\frac{x_t - x}{x}) \\ &= f_1(y, x)y\hat{y}_t + f_2(y, x)x\hat{x}_t, \end{aligned} \tag{2.2}$$

where

$$\hat{y}_t = \frac{y_t - y}{y}, \ \hat{x}_t = \frac{x_t - x}{x}.$$

Because the expansion in (2.2) is in terms of hatted variables, rather than deviations, it is referred to as a log-linear expansion. Now, we have a simple way to determine how the endogenous variable varies with the exogenous variable. Set simply solve $F(y_t, x_t) = 0$:

$$\hat{y}_t = -\frac{f_2(y, x)x}{f_1(y, x)y}\hat{x}_t.$$

6. **Log-linearizing the model.** The equations of the model are given by (2.1). As in the previous example, we have a solution for these equations at steady state (recall question 5). To analyze what happens to the endogenous variables, u_t and k_t, as the shock varies, we replace the equations in (2.1) with their log-linear expansion about the steady state. The log-linearized expansion of the first equation in (2.1) is:

$$v_{u,k}(k, u, \varepsilon)k\hat{k}_t + v_{u,u}(k, u, \varepsilon)u\hat{u}_t + v_{u,\varepsilon}(k, u, \varepsilon)\varepsilon\hat{\varepsilon}_t = 0,$$

or,

$$V_1^u \hat{k}_t + V_2^u \hat{u}_t + V_3^u \hat{\varepsilon}_t = 0, \tag{2.3}$$

where $V_1^u = v_{u,k}(k,u,\varepsilon)k$, $V_2^u = v_{u,u}(k,u,\varepsilon)u$, $V_3^u = v_{u,\varepsilon}(k,u,\varepsilon)\varepsilon$, and $v_{u,k}(k,u,\varepsilon)$ is the partial derivative of $v_u(k_t,u_t,\varepsilon_t)$ with respect to k_t, evaluated at $k_t = k$, $u_t = 1$, $\varepsilon_t = \varepsilon$. The objects, $v_{u,u}(k,u,\varepsilon)$ and $v_{u,\varepsilon}(k,u,\varepsilon)$ are defined similarly.

(a) Derive expressions, in terms of the model parameters, for V_1^u, V_2^u and V_3^u. Enter your formulas into MATLAB in the neighborhood of line 24 in hmk1.m.

(b) An alternative way to obtain V_1^u, V_2^u and V_3^u is to use numerical differentiation. The MATLAB program, derivative.m, computes a derivative, $f'(x)$, using the following formula, $f'(x) = (f(x+x_\varepsilon) - f(x))/x_\varepsilon$, where x_ε is small. Use this program to numerically differentiate v_u and compare the results with those based on your formulas. The program, derivative.m, requires that you supply it with the equation you want to differentiate. The first line of the program defining the equation must be as follows: function [eul] = Vu(x, σ_a, α, b). Here, x is a vector containing the variables being differentiated and the other three objects must contain the numerical values of the parameters, σ_a, α, b. The MATLAB call to derivative.m must have the form, [QQ]=derivative(x,'Vu', σ_a, α, b), where x is a vector containing the values of k_t, u_t, ε_t at the point were the derivative is to be evaluated. Thus, $x = (k, 1, \varepsilon)$ for this example. The elements of the vector QQ contain the derivatives of the function defined by Vu, with respect to the elements in x. You will be provided with derivative.m and a basic shell of a Vu.m for use in answering this question.

Compare the values of V_1^u, V_2^u and V_3^u that you obtained by numerical differentiation with the values you obtained using your formulas in (a).

(c) Now, log-linearly expand the dynamic Euler equation. As before, you have to supply the Euler equation to be differentiated (the shell of a MATLAB code, Vk.m, will be supplied to you for this purpose). Call the result:

$$E_t\left[V_1^k \hat{k}_t + V_2^k \hat{k}_{t+1} + V_3^k \hat{k}_{t+2} + V_4^k \hat{u}_t + V_5^k \hat{u}_{t+1} + V_6^k \hat{\varepsilon}_t + V_7^k \hat{\varepsilon}_{t+1}\right] = 0. \tag{2.4}$$

(Hint: it is best to evaluate the intertemporal Euler equation in two steps - first compute C_t, C_{t+1} and R_{t+1}^k and then construct the equation using these three variables. In particular, use the period t resource constraint to define C_t as a function of k_t, k_{t+1}, ε_t and u_t; the $t+1$ resource constraint to define C_{t+1} as a function of k_{t+1}, k_{t+2}, ε_{t+1} and u_{t+1}. Also, compute the period $t+1$ rate of return on capital, R_{t+1}^k as a function of u_{t+1}, k_{t+1}, , ε_{t+1}. Then, the intertemporal Euler equation is $-u'(C_t)+\beta u'(C_{t+1})R_{t+1}^k$, where $u'(C_t)=C_t^{-\sigma}$.) Display the values of V_i^k, $i=1,...,7$.

7. **Solving the Log-Linearized Model.** We have replaced equations, (2.1) with (2.3) and (2.4). The idea now is to find a 'solution' for them, in a sense to be defined precisely below. In the mean time, we make life simpler by using (2.3) to substitute out for $\hat{u}_t$ in (2.4). Solve (2.3) for $\hat{u}_t$:

$$\hat{u}_t=-\frac{V_1^u}{V_2^u}\hat{k}_t-\frac{V_3^u}{V_2^u}\hat{\varepsilon}_t. \tag{2.5}$$

Substitute this into (2.4),

$$\begin{aligned} E_t[V_1^k\hat{k}_t+V_2^k\hat{k}_{t+1}+V_3^k\hat{k}_{t+2}+V_4^k\left(-\frac{V_1^u}{V_2^u}\hat{k}_t-\frac{V_3^u}{V_2^u}\hat{\varepsilon}_t\right) & \\ +V_5^k\left(-\frac{V_1^u}{V_2^u}\hat{k}_{t+1}-\frac{V_3^u}{V_2^u}\hat{\varepsilon}_{t+1}\right)+V_6^k\hat{\varepsilon}_t+V_7^k\hat{\varepsilon}_{t+1}] &= 0, \end{aligned}$$

or,

$$E_t\left[\alpha_0\hat{k}_{t+2}+\alpha_1\hat{k}_{t+1}+\alpha_2\hat{k}_t+\beta_0\hat{\varepsilon}_{t+1}+\beta_1\hat{\varepsilon}_t\right]=0, \tag{2.6}$$

where

$$\begin{aligned} \alpha_2 &= V_1^k-V_4^k\frac{V_1^u}{V_2^u} \\ \alpha_1 &= V_2^k-V_5^k\frac{V_1^u}{V_2^u} \\ \alpha_0 &= V_3^k \\ \beta_1 &= V_6^k-V_4^k\frac{V_3^u}{V_2^u} \\ \beta_0 &= V_7^k-V_5^k\frac{V_3^u}{V_2^u}. \end{aligned}$$

We can now apply the undetermined coefficients method to find a solution. We posit that capital evolves as follows:

$$\hat{k}_{t+1}=A\hat{k}_t+B\hat{\varepsilon}_t, \tag{2.7}$$

where A and B are pinned down by the requirement that (2.6) is satisfied for all possible values of $\hat{k}_t$ and $\hat{\varepsilon}_t$. Note:

$$\begin{aligned} \hat{k}_{t+2} &= A\hat{k}_{t+1} + B\hat{\varepsilon}_{t+1} \\ &= A^2\hat{k}_t + (AB + B\rho)\,\hat{\varepsilon}_t + Be_{t+1} \end{aligned} \tag{2.8}$$

Substituting (2.8) and (2.7) into (2.6), we obtain:

$$E_t\left[\alpha_2\hat{k}_t + \alpha_1\left(A\hat{k}_t + B\hat{\varepsilon}_t\right) + \alpha_0\left(A^2\hat{k}_t + (AB + B\rho)\,\hat{\varepsilon}_t + Be_{t+1}\right) + \beta_1\hat{\varepsilon}_t + \beta_0\rho\hat{\varepsilon}_t + \beta_0 e_{t+1}\right] = 0$$

Noting that $E_t e_{t+1} = 0$ and collecting terms, the previous expression reduces to:

$$\alpha(A)\hat{k}_t + F\hat{\varepsilon}_t = 0,$$

where

$$\begin{aligned} \alpha(A) &= \alpha_0 A^2 + \alpha_1 A + \alpha_2 \\ F &= \alpha_1 B + \alpha_0\,(AB + B\rho) + \beta_1 + \beta_0\rho. \end{aligned} \tag{2.9}$$

So, A and B are pinned down by the requirements $\alpha(A) = 0$ and $F = 0$. To solve for A and B, first solve for A and then for B. Note that in general there are two solutions to $\alpha(A) = 0$. In practice it only makes sense to select a solution if it is less than unity in absolute value. For a value of A larger than unity in absolute value, the solution, (2.7) predicts that k_t will diverge away from its steady state (i.e., $\hat{k}_t$ will get very large), when the linear approximations are certain to not be valid. In some circumstances (for example, when $\sigma_a = \infty$, so there is no variable capital utilization), there exist theorems which indicate that there will be only one eigenvalue less than unity in absolute value, and that that one is the correct one to choose in approximating the true solution (see Stokey and Lucas, Chapter 6).

The value of A that we seek, satisfies

$$A = \frac{1}{\alpha_0}\left(-\frac{1}{2}\alpha_1 \pm \frac{1}{2}\sqrt{-4\alpha_0\alpha_2 + \alpha_1^2}\right),$$

and the property that it is less than unity in absolute value. (In this model, it can be proved that the two values of A, say A_1 and A_2, are both real and positive and satisfy $A_1 \leq 1$ and $A_2 \geq 1/\sqrt{\beta}$.) Then, solve for B from $F = 0$:

$$B = -\frac{\beta_1 + \beta_0\rho}{\alpha_1 + \alpha_0\,(A + \rho)}.$$

Enter into the indicated place in hmk1.m the formulas for α_0, α_1, α_2, β_0, β_1, A, B, and display the values of these variables.

8. **Interpretation of A ('speed of adjustment')**. Suppose there is no uncertainty, so that $\hat{\varepsilon}_t = 0$ for all t. We will consider how much time it takes for the economy to return to steady state, in case it starts in period 0 away from steady state. Note that for a given $\hat{k}_0$, $\hat{k}_1 = A\hat{k}_0$. Similarly, $\hat{k}_t = A^t\hat{k}_0$ for $t = 1, 2, 3, ...$. Literally, we must wait for $t = \infty$ before $\hat{k}_t = 0$ and the system is back to steady state. However, for large enough t, $\hat{k}_t$ is quite close. This motivates asking 'how long does it take to close 90% of the gap between the initial stock of capital and the steady state stock of capital'. That is, what value does t have to take on so that

$$\frac{\hat{k}_t}{\hat{k}_0} \equiv \frac{\frac{k_t - k}{k}}{\frac{k_0 - k}{k}} = \frac{k_t - k}{k_0 - k} = 0.10.$$

To determine the value of t, call it t^*, for which this holds, simply substitute in $A^{t^*}\hat{k}_0$ for $\hat{k}_{t^*}$ and solve for t^* :

$$\begin{aligned} A^{t^*} &= 0.10 \\ &\rightarrow t^* = \frac{\log 0.10}{\log A}. \end{aligned}$$

The speed of adjustment in the model, which is a function of A only, is the outcome of the interplay between two forces. Consider the case when the initial capital stock is low, so that $\hat{k}_0 < 0$. The speed of adjustment in this case corresponds to the amount of time it takes for the capital stock to grow back up to its steady state value. The incentive to save and grow fast is provided by the rate of return on capital. Note that, holding u_t constant, a lower capital stock raises the marginal product of capital. The greater is the curvature in the production function, (i.e., the smaller is α or the larger is δ), the greater is the incentive to grow quickly back to steady state when $\hat{k}_0 < 0$. Now, the economy could in principle grow back to steady state very quickly, simply by setting consumption to zero long enough. This is obviously not desirable, because it would make the marginal utility of consumption too high in the early periods. The greater is the curvature in the utility function (i.e., the larger is σ), the stronger is this force slowing the return to steady state.

(a) Compute the speed of adjustment (in annual units, so report $t^*/4$) for the benchmark parameter values. Also, consider 3 single-parameter deviations from the benchmark values. In the first case, set $\delta = 0$, and leave the other parameters at their benchmark values. In the second case, set $\alpha = 0.10$ and leave the other parameters at their benchmark values and in the third case set $\sigma = 4$. Are the results for speed of adjustment consistent with the intuition outlined above? (The validity of the intuition can be established formally by suitably applying the analysis in Stokey and Lucas, Chapter 6.)

(b) Set $\sigma_a = 10,000$, and leave the other parameter values at their benchmark settings. What happens to the speed of adjustment? For fun, also try the lowest value of σ_a you can get away with, without the program crashing (I managed to go to $\sigma_a = 0.00001$, when $t^* = 746,611$ years!) Do the results conform with intuition? (Hint: recall your answer to question 4).

9. **Other variables.** Given A and B, we can compute the response of the stock of capital to shocks in $\hat{\varepsilon}_t$, as well as to off-steady state initial conditions in the capital stock. Given the capital stock and $\hat{\varepsilon}_t$, we can also compute what happens to capital utilization using (2.5).

(a) Consumption. This can be obtained using the resource constraint, (1.2), evaluated as a strict equality and using the decision rule for utilization. The resource constraint is nonlinear in terms of the variables that are available, i.e., $\hat{k}_t$, $\hat{u}_t$ and $\hat{\varepsilon}_t$. To see this, note that $k_t = \left(\hat{k}_t + 1\right) k$ and $u_t = \hat{u}_t + 1$, while the production function, (1.3), is a non-linear function of k_t and u_t. Although C_t could be evaluated using the nonlinear function, for some purposes it is useful to have an expression that relates $\hat{C}_t$ linearly to $\hat{k}_t$ and $\hat{\varepsilon}_t$. This linear expression is obtained by taking a log-linear approximation of the resource constraint about the steady state and using (2.7). Write expression for consumption as follows:

$$\hat{C}_t = C_\varepsilon \hat{\varepsilon}_t + C_k \hat{k}_t. \tag{2.10}$$

(b) Investment. Compute the log-linear expression relating $\hat{I}_t^k$ to $\hat{k}_t$ and $\hat{\varepsilon}_t$:

$$\hat{I}_t^k = I_k \hat{k}_t + I_\varepsilon \hat{\varepsilon}_t \tag{2.11}$$

Note that if A is large enough (say because σ_a is very close to zero) then $\hat{I}_t^k$ is positive when the capital stock is above its steady state. This may at first seem inconsistent with stability property of our system: when the capital stock is above steady state it comes down, and when it is below steady state it goes up. Apparently, investment can be positive when the capital stock is above steady state! How is this consistent with stability?

(c) Gross output. A measure of GNP in this economy is the total output of goods, Y_t (see (1.3)), net of maintenance costs of capital, I_t^u (see (1.6)). Call GNP Y_t^{gnp}, so that

$$Y_t^{gnp} = Y_t - I_t^u.$$

Compute the log-linear expression relating $\hat{Y}_t^{gnp}$ to $\hat{k}_t$ and $\hat{\varepsilon}_t$:

$$\hat{Y}_t^{gnp} = Y_k \hat{k}_t + Y_\varepsilon \hat{\varepsilon}_t \tag{2.12}$$

(d) Compute C_ε, C_k, I_k, I_ε at the benchmark parameter values. There is an obvious consistency condition that must hold across these numbers, due to the resource constraint, $C_t + I_t^k = Y_t^{gnp}$.

$$Y_k = (C/Y)C_k + (\delta k/Y)I_k,\ Y_\varepsilon = (C/Y)C_\varepsilon + (\delta k/Y)I_\varepsilon.$$

It is worthwhile verifying that these consistency conditions are satisfied.

10. **Model Simulation.** We can now study the dynamic response of the economy to a shock that occurs in period $t = 1$, when $\hat{k}_1$ takes on some given value. Suppose $e_1 = .01$, so that there is a one percent innovation in $\hat{\varepsilon}_1$. Then, $\hat{\varepsilon}_t = \rho\hat{\varepsilon}_{t-1}$, $t = 2, 3, ...$. Also, $\hat{k}_{t+1} = A\hat{k}_t + B\hat{\varepsilon}_t$, $t = 1, 2, ...$. Finally, the other variables of interest can be computed from (2.5) and (2.10)-(2.12). These variables can then be graphed as is. In this case, the graphs are of the percent (if multiplied by 100) deviation of the variables from their steady state values. Alternatively, the actual values of the variables can be graphed. That is, if $\hat{x}_t$ is a variable of interest, then its level can be obtained from:

$$x_t = (\hat{x}_t + 1)\, x. \tag{2.13}$$

An alternative transformation is also sometimes used, to convert to levels. Note from the previous expression, that $\log x_t = \log(\hat{x}_t + 1) + \log x$. It is also the case that when $\hat{x}_t$ is close to zero (and the approximations we use are strictly only valid in this case anyway), then $\log(\hat{x}_t + 1) \approx \hat{x}_t$. Using this approximation, we see

$$\hat{x}_t = \log \frac{x_t}{x},$$

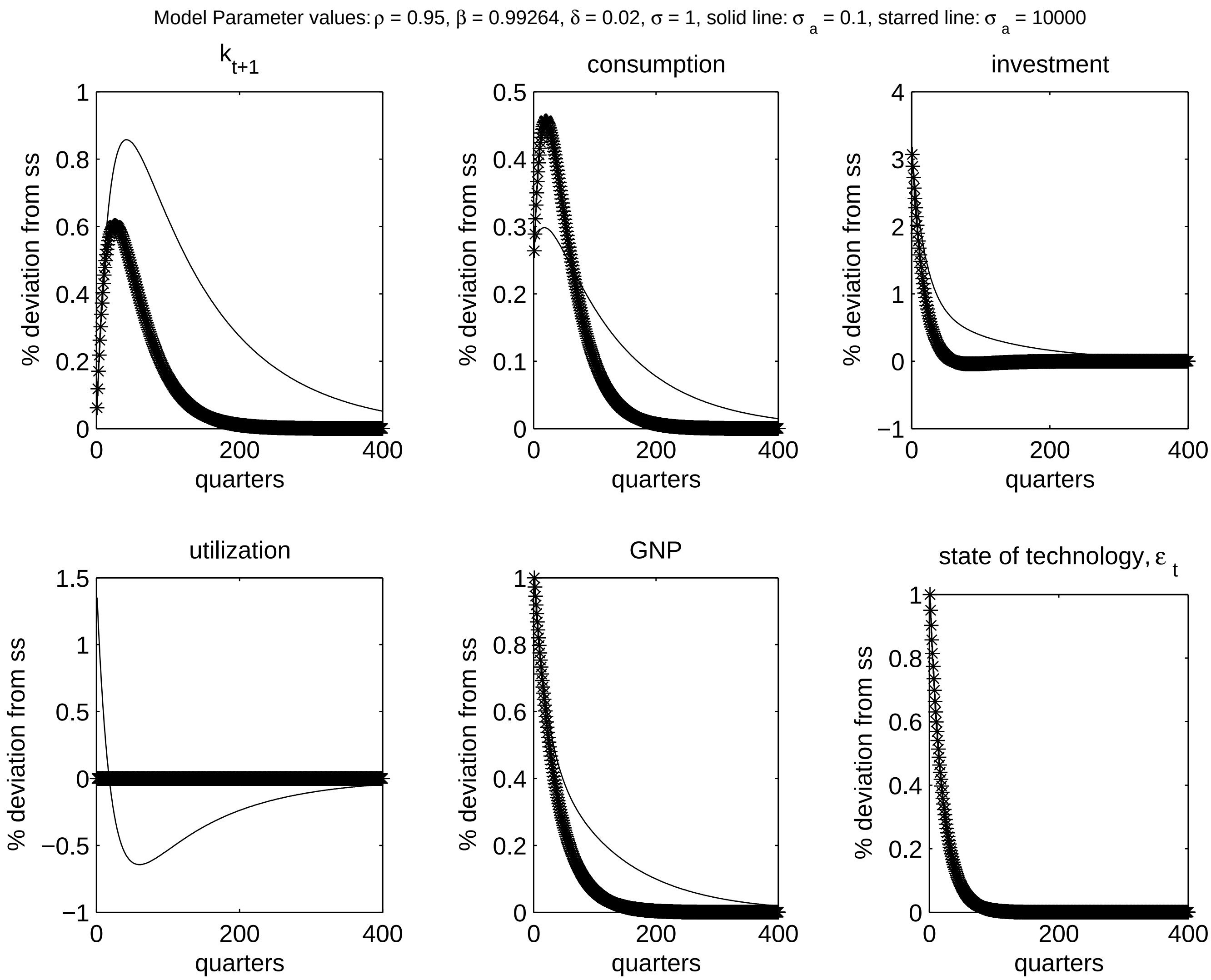
or,

$$x_t = x \exp\left(\hat{x}_t\right). \tag{2.14}$$

It is not clear which of the two, (2.13) or (2.14), is the better approximation.

Model simulation is quite straightforward, and so you are not asked to write the code to do simulations for this homework. Instead, you can use the MATLAB routine that has been provided, hmk1answer.m, to do the simulation. The start of this program looks just like hmk1.m, and this is where the parameter values are input, as well as T, the number of observations you'd like to simulate (the code that you are asked to enter into hmk1.m in the questions above has been entered into solvemodel.m, which is called by hmk1answer.m.) The program, hmk1.m calls solvemodel.m, which solves the model and returns $\hat{C}_t$, $\hat{I}_t^k$, $\hat{u}_t$, $\hat{k}_{t+1}$, $\hat{Y}_t^{gnp}$, $\hat{\varepsilon}_t$, $t = 1, 2, ..., T$. In the calculations, $e_1 = 0.01$, and $e_t = 0$ for $t \geq 2$. The program, solvemodel.m, is executed twice, once with $\sigma_a = 0.1$ and once with $\sigma_a = 10,000$. The output of both series was graphed by plotout.m, so that you can see the impact of σ_a on the transmission of a technology shock.

When $\sigma_a = 10,000$ capacity utilization is essentially fixed, because it is prohibitively expensive to vary u_t. In the case, $\sigma_a = 0.1$, u_t is relatively inexpensive to vary. What is graphed in the following figure is, in each case, the hatted variable, times 100. Note in the figure that the high value of $\hat{\varepsilon}_t$ in the early periods leads to an initial high level of utilization. Eventually, as the capital stock grows, it dominates in the utilization decision, and utilization starts to go negative. The higher rate of return associated with low capital utilization then encourages more investment.



You may find it interesting to experiment with hmk1answer.m, to see how different settings of the parameters affect the transmission of technology shocks.

11. **Unit Roots.** Often, it is of interest to specify that the technology shock have a unit root. We can accommodate this by supposing that the state of technology, now labelled z_t, evolves as follows:

$$z_t = z_{t-1} \exp(\varepsilon_t), \tag{2.15}$$

where

$$\varepsilon_t = (1-\rho)\varepsilon + \rho\varepsilon_{t-1} + e_t.$$

We replace the production function, (1.3), with:

$$Y_t = (u_t k_t)^{\alpha} z_t^{1-\alpha}.$$

In this new specification, z_t is the state of technology. Note that technology now has a unit root. That is,

$$\log z_t - \log z_{t-1} = \varepsilon_t,$$

where ε_t is a first order autoregressive process. With this specification, a shock to e_t has a much more powerful impact on technology than it did before. Now, e_t drives z_t (and, $\log z_t$) up permanently. A unit innovation in e_t drives up $\log z_t$ by the same amount, $\log z_{t+1}$ by $(1+\rho)\, e_t$, and so forth. The eventual impact of a unit jump in e_t is to raise $\log z_t$ by $1/(1-\rho)$ permanently. To see how big this can be, suppose $\rho = 0.90$. In this case a unit shock to $\log z_t$ eventually leads to a ten-fold rise in $\log z_t$.

In the new specification of the model, we use the same utility function and resource constraint as before, (1.1) and (1.2), respectively. We also use the equations pertaining to investment and capital utilization, (1.4)-(1.7).

With the new setup, the intertemporal Euler equation is:

$$E_t \left\{ C_t^{-\sigma} - \beta C_{t+1}^{-\sigma} \left[\alpha u_{t+1}^{\alpha} k_{t+1}^{\alpha-1} z_{t+1}^{1-\alpha} + (1 - \delta - a\left(u_{t+1}\right)) \right] \right\} = 0,$$

and the utilization first order condition is:

$$\alpha u_t^{\alpha-1} k_t^{\alpha} z_t^{1-\alpha} = a'\left(u_t\right) k_t.$$

We want to apply the style of analysis used in the previous model economy. However, that analysis required that the variables converge to a steady state. The problem is that in this model economy, the variables do not converge to a steady state. Instead, when e_t is kept fixed at its unconditional mean, ε_t converges $\varepsilon = E\varepsilon_t$ and z_t grows perpetually at $\varepsilon \times 100$ percent per period. All the other variables, C_t, Y_t, k_{t+1}, etc., then also grow at $\varepsilon \times 100$ per period (this will be established formally below). So, the methods developed previously will not work for C_t, Y_t, k_{t+1}. However, it turns out that if we scale these variables, then the method does apply to the scaled variables. In particular, consider the following change of variables:

$$c_t = \frac{C_t}{z_t},\ \tilde{k}_{t+1} = \frac{k_{t+1}}{z_t},\ y_t = \frac{Y_t}{z_t}.$$

Rewrite the intertemporal Euler equation, replacing C_t with $z_t c_t$ and k_{t+1} with $\tilde{k}_{t+1} z_t$, multiply both sides of the result by z_t^{σ}, and take into account (2.15). Then,

$$E_t\left\{c_t^{-\sigma}-\beta\exp(-\sigma\varepsilon_{t+1})c_{t+1}^{-\sigma}\left[\alpha u_{t+1}^{\alpha}\left(\exp(-\varepsilon_{t+1})\tilde{k}_{t+1}\right)^{\alpha-1}+\left(1-\delta-a\left(u_{t+1}\right)\right)\right]\right\}=0. \tag{2.16}$$

The Euler equation for u_t, in terms of scaled variables, is:

$$\alpha u_t^{\alpha-1}\exp(-\alpha\varepsilon_t)\tilde{k}_t^{\alpha}-a'\left(u_t\right)\exp(-\varepsilon_t)\tilde{k}_t=0. \tag{2.17}$$

Rewriting the resource constraint in a similar way, we obtain:

$$c_t=\left(u_t\exp(-\varepsilon_t)\tilde{k}_t\right)^{\alpha}-\left(\tilde{k}_{t+1}-(1-\delta)\exp(-\varepsilon_t)\tilde{k}_t\right)-a\left(u_t\right)\exp(-\varepsilon_t)\tilde{k}_t. \tag{2.18}$$

The previous three equations characterize equilibrium for utilization, as well as for scaled consumption and capital accumulation. The variables u_t, $\tilde{k}_t$ and c_t, have well-defined steady states. This can be verified by noting that if $\varepsilon_t=\varepsilon$, for all t, the three equations can be solved for u, $\tilde{k}$ and c (this proves an assertion made previously). The objects, $\tilde{k}$ and c, correspond to steady state growth paths for k_t and $c_t: k_t=z_t\tilde{k}$, $C_t=cz_t$. Along a steady state growth path, u_t is constant (actually, $u=1$ if we restrict the function, (1.7), appropriately).

To analyze the system when it is stochastically perturbed from its steady state growth path, we replace the nonlinear euler equations in (2.16)-(2.18) by their expansion about the values of the variables in steady state. One small difference arises from the specification of the technology shock, which causes the exponential of ε_t to appear in the scaled equations. This leads us to work with a linear expansion of these variables in terms of $\tilde{\varepsilon}_t=\varepsilon_t-\varepsilon$. For example, denote the function, (2.17) by $v_u(\tilde{k}_t,\varepsilon_t,u_t)$. Then, the expansion we work with is:

$$v_u(\tilde{k}_t,\varepsilon_t,u_t)\simeq v_u(\tilde{k},\varepsilon,u)+v_{u,\tilde{k}}(\tilde{k},\varepsilon,u)\tilde{k}\hat{\tilde{k}}_t+v_{u,\varepsilon}(\tilde{k},\varepsilon,u)\tilde{\varepsilon}_t++v_{u,u}(\tilde{k},\varepsilon,u)u\hat{u}_t,$$

where $v_{u,x}$ means the partial derivative of the function, v_u, with respect to x. Note that in terms of ε_t, this is the usual Taylor series expansion. In terms of capital and utilization, we continue to work with the log-linear expansion.

After linearizing (2.16)-(2.18), we can substitute out for consumption and utilization, to obtain a single dynamic equation in terms of capital and the ε_t shock only. The undetermined coefficient method can then be applied again to obtain a solution of the following form:

$$\hat{\tilde{k}}_{t+1}=A\hat{\tilde{k}}_t+B\tilde{\varepsilon}_t,$$

where A and B are pinned down by the analogs to (2.9). As before, we solve for consumption and investment as follows:

$$\begin{aligned}\hat{c}_t &= c_k\hat{\tilde{k}}_t + c_\varepsilon\tilde{\varepsilon}_t \\ \hat{\imath}_t^k &= i_k\hat{\tilde{k}}_t + i_\varepsilon\tilde{\varepsilon}_t,\end{aligned}$$

where $i_t^k = I_t^k/z_t$. Simulations proceed as before, though with a twist. The twist arises because the variables being simulated are not the actual variables of interest, but (apart from utilization) they are scaled variables. So, after simulating $\hat{\tilde{k}}_{t+1}$, $t = 1, 2, ..., T$, this variable must be converted to unscaled form by multiplication, $z_t\hat{\tilde{k}}_{t+1}$. Similarly, for investment and consumption. Utilization, of course, need not be 'unscaled' since it is not scaled in the first place.

Sometimes, it is not the unscaled variable that is required. In the case of an impulse response function, it is assumed that the shock hits the economy when it is on a steady state growth path. In practice, the impulse response function is defined as the path a variable takes in response to a shock, relative to what its path would have been if there had been no shock. In our analysis of the stationary economy, this definition of an impulse response function led us to graph variables like $\hat{C}_t$. Now, we wish to display unscaled consumption along the shocked equilibrium path, $C_t = (\hat{c}_t + 1)\, cz_t$, relative to what it would have been if no shock had occurred. Let $\tilde{z}_t$ denote the path that z_t would have taken in the absence of a shock to e_t. Suppose that the shock occurs in period $t = 1$ and suppose the value of z_t in period $t = 0$ is z_0. Then,

$$\begin{aligned}\tilde{z}_t &= \exp(t\varepsilon)z_0 \\ z_t &= \exp(\varepsilon_t)z_{t-1} \\ &= \exp(\varepsilon_t + \varepsilon_{t-1})z_{t-2} \\ &= ... \\ &= \exp(\varepsilon_t + \varepsilon_{t-1} + ... + \varepsilon_1)z_0\end{aligned}$$

Note that z_0.is the same for the shocked and the unshocked paths. If the shock had not occurred, then unscaled consumption would have been, say, $\tilde{C}_t$, where $\tilde{C}_t = c\tilde{z}_t$. Note,

$$\begin{aligned}\frac{C_t}{\tilde{C}_t} &= \frac{\exp(\varepsilon_t + \varepsilon_{t-1} + ... + \varepsilon_1)}{\exp(t\varepsilon)}(\hat{c}_t + 1) \\ &= \exp(\tilde{\varepsilon}_t + \tilde{\varepsilon}_{t-1} + ... + \tilde{\varepsilon}_1)(\hat{c}_t + 1).\end{aligned}$$

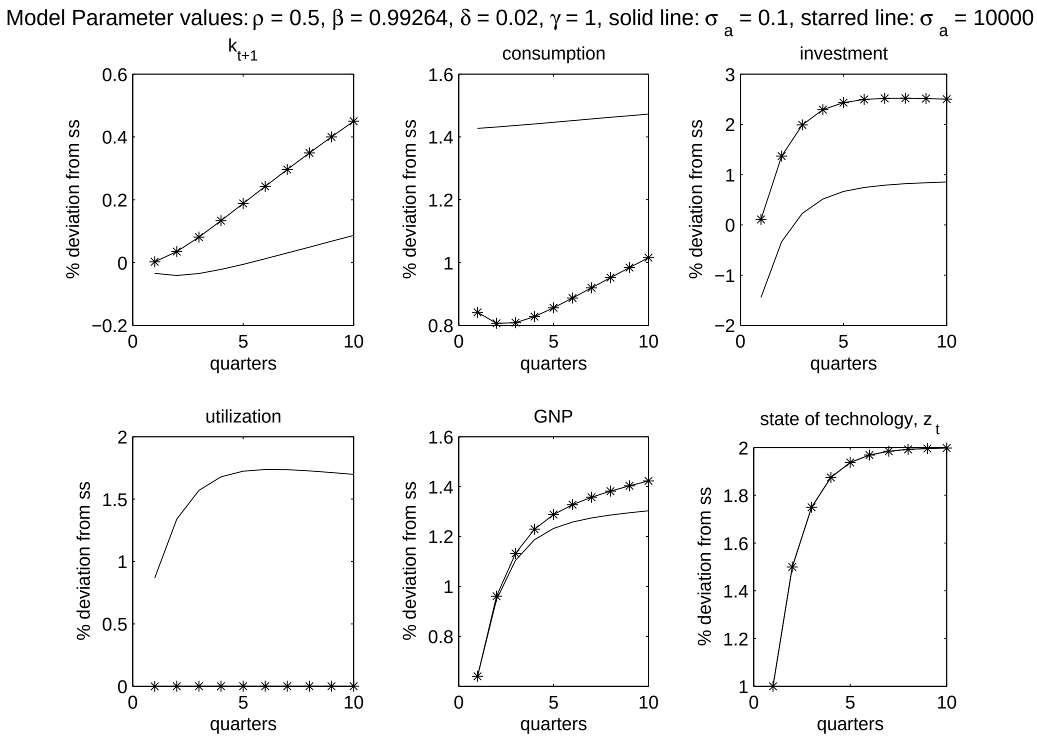
Then, the log deviation of shocked consumption from its unshocked path is:

$$\log \frac{C_t}{\tilde{C}_t} = \tilde{\varepsilon}_t + \tilde{\varepsilon}_{t-1} + ... + \tilde{\varepsilon}_1 + \hat{c}_t,$$

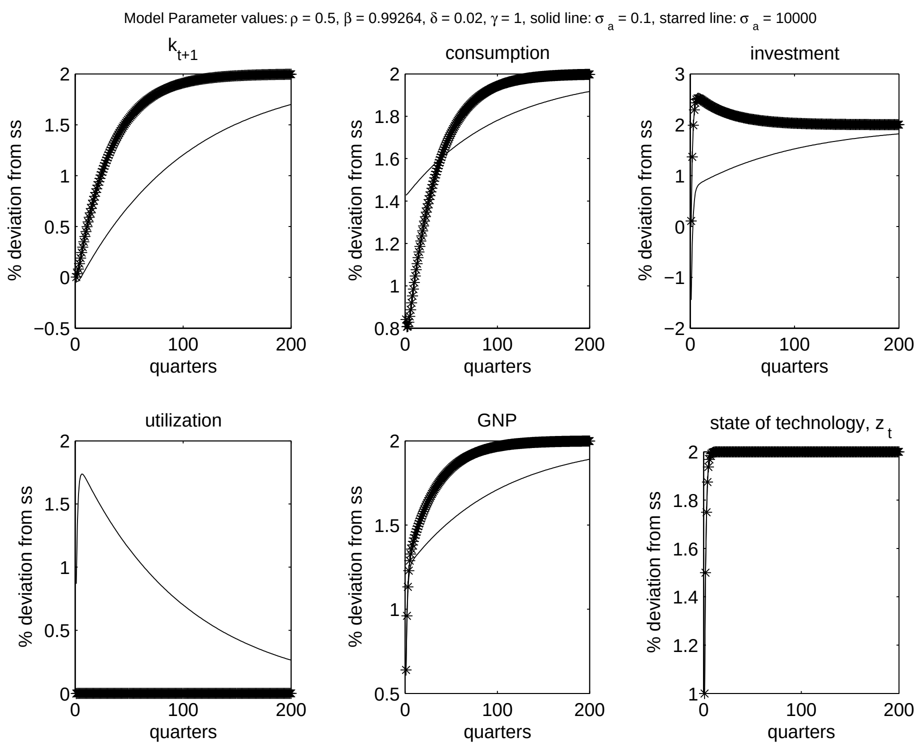
where we have used $\log(\hat{c}_t + 1) \simeq \hat{c}_t$. We conclude that if $\hat{x}_t$ is the solution of the log-linearized system for some variable that has been scaled, then the impulse response of that variable in period t is obtained simply by adding to it the cumulative sum of the $\tilde{\varepsilon}_t$'s.

These calculations for solving and simulating the model with the unit root shock have been programmed in MATLAB file solveneoclassicalunit.m. This program requires as input the model parameters, $(\beta, \alpha, \varepsilon, \delta, \sigma_a, \sigma, \rho)$. On output, the program produces A, B and the steady state (scaled) capital stock. (The program can also be used to repeat some of the calculations for the earlier part of this tutorial; with unit=1 the model with the unit root shock is analyzed, and with unit=0, the model with stationary shock is analyzed.) To call solveneoclassicalunit.m, run program neoclassical.m.

The following parameter values were selected: $\beta = 1.03^{-0.25}$, $\alpha = 0.36$, $\delta = 0.02$, $\rho = 0.5$, $\varepsilon = 0.015/4$, $\sigma = 1$. As before, the calculations were done for $\sigma_a = 0.1$ (see the solid line in the figure) and $\sigma_a = 10,000$ (see the starred line in the figure). The figure reports the response to a 1% shock to z_t (i.e., $e_1 = .01$ and $e_t = 0$ for $t > 1$). Note how the state of technology jumps one percent in the period of the shock, and eventually rises to 2 percent. In response to this, utilization jumps in the variable utilization case, and remains fixed at unity in the fixed case. Note how investment actually drops with variable utilization, while it rises with fixed utilization. The reason for this is simple. The innovation in technology is a signal that in the future there will be ample resources. This wealth effect exerts upward pressure on consumption, and downward pressure on investment. An effect working in the other direction stems from the fact that the jump in the state of technology raises the rate of return on capital. In the fixed utilization case, this rate of return effect dominates and investment rises. In the variable utilization case, utilization rises in response to the technology shock. This has the effect of reducing the rise in the rate of return of investment. Because the rate of return effect is weakened so much, the wealth effect dominates and this is why investment drops in the first few periods after the technology shock.



We know that consumption and investment must eventually rise permanently by 2 percent, and utilization must eventually return to unity. To see that it actually does happen, is useful to simulate the model for longer periods. This is done in the following figure, which displays results for $T = 200$. Note how much more time the model with variable capital utilization takes to converge to steady state. The fact that high capital utilization in the early periods slows the response of investment, means that the capital stock takes much longer to rise up to its new steady state growth path



Question: run the program, neoclassical.m, with $\rho = 0.99$. Note that now investment is below steady state for 135 periods in the variable capital utilization case. It is also negative for 13 periods in the fixed utilization case. Is this consistent with the intuition outlined above? Now run the program with $\sigma = 10$, and note that it takes much longer for the system to converge to its new steady state growth path after a shock. Is this consistent with intuition? You may want to play with other parameter values, to build intuition.

References

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[2] Christiano, Lawrence J., 2002, 'Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients', Computational Economics, October, Volume 20, Issue 1-2.