Chapter 1

Preliminaries

This chapter introduces interest rates and growth rates. The two topics are closely related, so we treat them together. The concepts discussed here are not in Barro, but they will help you understand the graphs and statistics that he uses throughout his book.

1.1 Compound Interest

We begin with some common terms and calculations from the realm of fixed-income investments. The amount of the investment is called the *principal*. The "fixed-income" from the investments is called *interest*. The interest per unit of principal per unit of time is called the *interest rate*. Most commonly, interest rates are quoted in dollars per year per dollar of principal. These units can be written: $\$/(y\$)$. The dollar units cancel, so this interest rate has units of one over years. Similarly, if the interest rate is apples per day per apple borrowed, the apple units will cancel, and the units of the interest rate will be one over days. In general, the units of an interest rate are one over some unit of time.

When the unit of time is a year, we say that an interest rate is an *annual* interest rate. If the unit of time is not mentioned, then it will almost always be an annual interest rate. Interest rates that are quoted in some specific unit of time can be converted to any other unit of time via a simple linear transformation. For example, a daily interest rate of x% corresponds to an annual interest rate of (365)(x)%.[1] (See Exercise 1.1 for an example.)

We use P for the principal of a fixed-income investment and R for the annual interest rate. Under *simple interest* the interest is earned on the amount of the principal only. In this case,

[1]You may be wondering about leap years. These are handled according to any of a number of conventions. For example, some interest rates are quoted using 360 days as a year; others use 365; still others use 365.25.

after n years the value of the investment will be:

$$V_s(n) = RPn + P. \tag{1.1}$$

For example, suppose you invest $5,000 at a 4.5% simple annual interest rate. After two years the value of your investment will be:

$$V_s(2) = (0.045)(\$5,000)(2) + \$5,000 = \$5,450.$$

It is much more common for interest to be compounded annually. In this case, at the end of each year, that year's interest will be added to the principal, so the investment will earn interest on the interest. The first year will be just like simple interest, since none of the interest will yet be compounded. Accordingly, the the value after the first year will be:

$$V_a(1) = RP + P = (1 + R)P.$$

After the second year, the value will be:

$$V_a(2) = RV_a(1) + V_a(1) = R(1 + R)P + (1 + R)P = (1 + R)^2P.$$

Similarly, after n years, the value will be:

$$V_a(n) = (1 + R)^n P. \tag{1.2}$$

Of course, this formula works only an integral numbers of years. For non-integral numbers, you round down to the nearest integral year n, compute $V_a(n)$, and use that in the simple-interest formula (1.1) for the fraction of the last year. (See Exercise 1.6 for an example.)

Let's revisit our previous example. Once again, you invest $5,000 at a 4.5% annual interest rate, but this time interest compounds annually. After two years the value of your investment will be:

$$V_a(2) = (1 + 0.045)^2(\$5,000) = \$5,460.13.$$

(Here and throughout, dollar amounts are rounded to the nearest cent.) Notice that the investment is worth less under simple interest than under compound interest, since under compounding you earn about $10 of interest on the first year's interest.

The above reasoning for compounding annually applies to compounding more frequently. The only catch is that the interest rate needs to be quoted in terms of the same time interval as the compounding. If R is an annual interest rate, and interest is to compound t times per year, then the value of an investment after n years will be:

$$V_t(n) = \left(1 + \frac{R}{t}\right)^{tn} P.$$

We return to our example again, this time supposing that interest compounds daily. After two years, the value will be:

$$V_{365}(2) = \left(1 + \frac{0.045}{365}\right)^{(365)(2)} (\$5,000) = \$5,470.84.$$

As we compound more and more frequently, we arrive at the expression for continuous compounding:

$$V_c(n) = \lim_{t\to\infty}\left(1+\frac{R}{t}\right)^{tn} P.$$

We can make this much more tractable by using the fact that:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x},$$

where e is Euler's constant. This gives us the following formula for continuous discounting:

$$V_c(n) = \lim_{t\to\infty}\left(1+\frac{R}{t}\right)^{tn} P = \left[\lim_{(t/R)\to\infty}\left(1+\frac{1}{(t/R)}\right)^{(t/R)}\right]^{Rn} P = e^{Rn}P. \tag{1.3}$$

We return to our example one last time, this time assuming continuous compounding. After two years, the value of the investment will be:

$$V_c(2) = e^{(0.045)(2)}(\$5,000) = \$5,470.87.$$

Again, notice how throughout these examples the value of the investment is greater the more often the interest compounds. Continuous compounding results in the highest value, but the returns to more-frequent compounding fall off fairly quickly. For example, the value is almost the same under daily versus continuous discounting.

1.2 Growth Rates

Economists are often interested in the *growth rates* of economic variables. You might read, "Real Gross Domestic Product grew at a 2.3% annual rate this quarter" or "Inflation is 4%" or "The world's population is growing 20% every decade." Each of these statements deals with a growth rate.

An interest rate is just the growth rate of the value of an asset, and all the terminology and formulae from the previous section apply to growth rates generally. For example, we can calculate simple annual growth rates and annual growth rates that are compounded annually or continuously.

Consider the following values for the Gross Domestic Product (GDP) of a hypothetical country:

Year	GDP
1991	\$100,000,000
1992	\$130,000,000
1993	\$135,000,000

The growth rate of GDP is just the interest rate that GDP would have had to earn if it were a fixed-income investment.

For example, the simple rate of growth of GDP between 1992 and 1993 is given by R in equation (1.1). Starting GDP is P, ending GDP is $V_s(n)$, and n is one year. Plugging all the numbers in, we get:

$$\$135K = (R)(\$130K)(1) + \$130K, \text{ so:}$$
$$R = \$135K/\$130K - 1 \approx 1.03846154 - 1 = 3.846154\%.$$

As another example, to calculate the annual rate of growth of GDP, compounded annually, between 1991 and 1993, we use equation (1.2). Starting GDP is P, ending GDP is $V_a(n)$, and n is two years. This gives us:

$$\$135K = (1 + R)^2(\$100K), \text{ so:}$$
$$R = (\$135K/\$100K)^{(0.5)} - 1 \approx 1.16189500 - 1 = 16.189500\%.$$

As a final example, we do the same calculation, but using continuous compounding. We just solve equation (1.3) for R. Starting GDP is P, ending GDP is $V_c(n)$, and n is two years.

$$\$135K = e^{2R}(\$100K), \text{ so:}$$
$$R = [\ln(\$135K) - \ln(\$100K)](0.5) \approx 0.15005230 = 15.15005230\%.$$

Economists generally prefer to use continuous compounding, for two reasons. First, under continuous compounding, computing the growth rate between two values of a series requires nothing more than taking the difference of their natural logarithms, as above.

This property is useful when graphing series. For example, consider some series that is given by $V(n) = V_0 e^{0.08n}$, which is depicted in Figure 1.1. By the equations above, we know that this series grows at an 8% continuous rate. Figure 1.2 depicts the natural logarithm of the same series, i.e., $\ln[V(n)] = \ln(V_0) + 0.08n$. From the equation, you can see that this new series is linear in n, and the slope (0.08) gives the growth rate. Whenever Barro labels the vertical axis of a graph with "Proportionate scale", he has graphed the natural logarithm of the underlying series. For an example, see Barro's Figure 1.1.

The second reason economists prefer continuous growth rates is that they have the following desirable property: if you compute the year-by-year continuous growth rates of a series and then take the average of those rates, the result is equal to the continuous growth rate over the entire interval.

For example, consider the hypothetical GDP numbers from above: $\$100K$, $\$130K$, and $\$135K$. The continuous growth rate between the first two is: $\ln(\$130K) - \ln(\$100K)$. The continuous growth rate between the second two is: $\ln(\$135K) - \ln(\$130K)$. The average of these two is:

$$\frac{\left[\ln(\$135K) - \ln(\$130K)\right] + \left[\ln(\$130K) - \ln(\$100K)\right]}{2}.$$

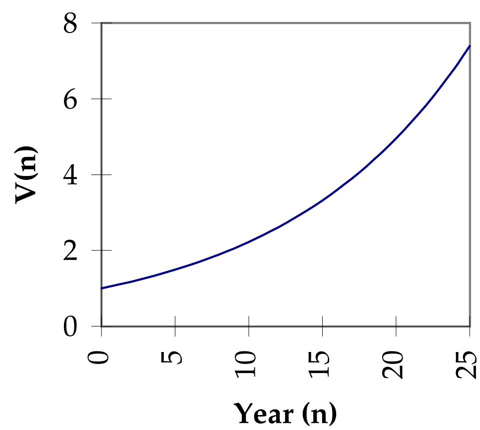

Figure 1.1: Exponential Growth

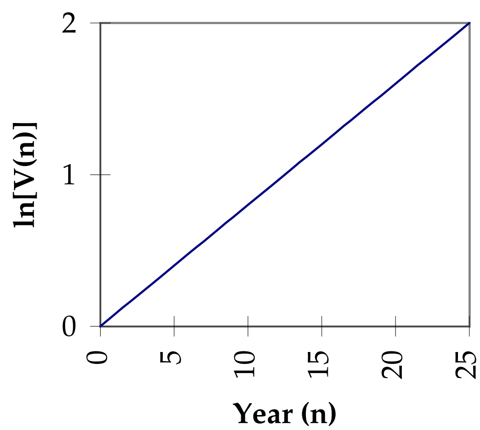

Figure 1.2: Log of Exp Growth

The two ln($\$130K$) terms cancel, leaving exactly the formula for the continuous growth rate between the first and third values, as we derived above.

If we carry out the same exercise under simple growth or annually compounded growth, we will find that the average of the individual growth rates will not equal the overall growth rate. For example, if GDP grows by 8% this year and 4% next year, both calculated using annual compounding, then the two-year growth rate will *not* be 6%. (You should verify that it will actually be 5.98%.) On the other hand, if the 8% and 4% numbers were calculated using continuous compounding, then the continuous growth rate over the two-year period would be 6%.

Exercises

Exercise 1.1 (Easy)
My credit card has an APR (annualized percentage rate) of 16.8%. What is the daily interest rate?

Exercise 1.2 (Easy)
My loan shark is asking for $25 in interest for a one-week loan of $1,000. What is that, as an annual interest rate? (Use 52 weeks per year.)

Exercise 1.3 (Moderate)
The Consumer Price Index (CPI) is a measure of the prices of goods that people buy. Bigger numbers for the index mean that things are more expensive. Here are the CPI numbers for four months of 1996 and 1997:

Variable	Definition
P	Principal (amount invested)
R	Nominal interest rate
n	Number of years invested
$V_s(n)$	Value after n years under simple interest
$V_a(n)$	Value after n years under annual compounding
$V_t(n)$	Value after n years when compounded t times per year
$V_c(n)$	Value after n years under continuous compounding
V_0	Initial value of the investment

Table 1.1: Notation for Chapter 1

Year	Mar	Jun	Sep	Dec
1996	155.7	156.7	157.8	158.6
1997	160.0	160.3	161.2	161.3

What is the growth rate of the CPI between June 1996 and September 1996? (Use a continuous growth rate and annualize your answer.)

Exercise 1.4 (Moderate)
Use the CPI data from the previous exercise to compute the growth rates in the CPI in the four quarters starting in March 1996 (i.e, Mar-Jun 1996, Jun-Sep 1996, etc.). (Use a continuous growth rate but do not annualize your answer.) Show that the sum of these four rates equals the (continuous) growth rate from March 1996 to March 1997.

Exercise 1.5 (Easy)
Real output of the United States will likely grow by about 2% over the first half of the next century. At that rate (of continuous growth), how long will it take for real output to double? Compare your exact answer with the approximation given by the "Rule of 72."[2]

Exercise 1.6 (Hard)
This morning you invest $10,000 at 6.5% interest that compounds annually. What is the first date on which you would have at least $15,000? (Quote the answer in terms of years + days from today. Interest accrues each night, but compounds only annually.)

Exercise 1.7 (Easy)
Suppose that 4.6 percent of the earth's forests are cleared each year. How long will it take

[2]The "Rule of 72" is as follows. If the interest rate on an investment is x percent, then it takes about $72/x$ years for the value of the investment to double.

for half our current forests to be cleared? (Use annual compounding and solve for the fewest number of whole years.)

Exercise 1.8 (Moderate)
World population was about 679 million in the year 1700 and about 954 million in 1800.

1. What was the annual growth rate of population between 1700 and 1800? (Use continuous compounding.)
2. Suppose that the human race began with Adam and Eve and that the annual growth rate between 1700 and 1800 prevailed in all years prior to 1700. About when must it have been that Adam and Eve were evicted from the Garden of Eden? (Hint: What was the population in that year?)

Exercise 1.9 (Moderate)
According to figures compiled by the World Bank, per capita real income in the U.S. was \$15,400 in 1984, while the corresponding figure for Japan was \$10,600. Between 1965 and 1984, per capita real income in the U.S. grew at an annual rate of 1.7 percent (using annual compounding), while the corresponding figure for Japan was 4.7 percent.

1. If these two growth rates remain constant at their 1965-84 levels, in what year will per capita real income be the same in these two countries? (Again, use annual compounding, and use hundredths of a year.)
2. What will be the common per capita real income of these two countries at that date?