Chapter 14

The Optimal Path of Government Debt

Up to this point we have assumed that the government must pay for all its spending each period. In reality, governments issue debt so as to spread their costs across several periods, just like households do. The path of governmental debt over time very often corresponds to events of major historical import, such as wars. For example, England has paid for her wars by issuing debt, resulting in debt peaks during the Seven Years War, the Napoleonic Wars, and especially World War I. Indeed, some economists argue that the sophistication of England's capital markets contributed to her eventual successes in the wars of the 17th and 18th centuries.

The U.S. and several European countries have run persistent peacetime deficits since about 1979. Quite a bit has been written in the popular press about the dire consequences of the ever-mounting debt. In this chapter we will not consider that a large debt may be inherently bad, instead we will treat the debt as a tool for use by a benevolent government. In the previous two chapters we saw that government spending may crowd out consumption and investment, and that government taxes may decrease labor supply and capital accumulation, but in this chapter we will have nothing bad to say about debt. In Chapter 18, however, we argue that under certain circumstances, large and persistent government deficits may be inflationary. In this chapter we will continue to ignore the price level and the ability of the government to raise revenue by printing money, so we will not be able to consider inflation directly.

We will begin by considering the government budget deficit and defining some terms. The reader should be thoroughly familiar with the terms defined there, as well as the historical paths of the debt, deficits, debt to GDP ratios and so on.

Next we will consider a very simple theory of the debt, that of *Barro-Ricardo Equivalence.*

Barro-Ricardo Equivalence is named for Robert Barro, at Harvard, and David Ricardo, the 19th-century economist. In this theory, the timing of government taxes and spending (and hence the path of the debt) do not matter. Only the present discounted value of these objects is important. We shall see that Barro-Ricardo Equivalence requires some strong assumptions. As we relax those assumptions, the timing of taxes begins to matter.

Requiring the government to use distortionary taxes is one way of breaking Barro-Ricardo Equivalence. In the final two sections of this chapter, we construct a fairly sophisticated theory of government debt based on precisely this assumption. This is known as the *Ramsey Optimal Tax problem* (or simply the Ramsey problem, for short). In the Ramsey problem the government has access only to a distortionary tax (in this case, an excise tax), and must raise a specific amount of revenue in the least-distortionary manner. In this example, the government will have to finance a war, modeled as a spike in planned government expenditures, with a period-by-period excise tax. By finding the optimal path of tax revenues, we can find the optimal path of government deficits and surpluses. This will provide us with a theory of government debt and deficits.

One of the features of the Ramsey model will be that both the government and the household will have access to a perfect loan market at a constant interest rate. This interest rate will not vary with the amount actually borrowed or lent, nor will it vary across time for other reasons. This is sometimes known as the "small open economy" equilibrium, but truthfully we are simply abstracting from the question of equilibrium entirely. No markets will clear in this example.

14.1 The Government Budget Constraint

Let $\mathcal{T}_t$ be the real revenue raised by the government in period t, let G_t be real government spending in period t (including all transfer payments) and let B_t^g be the real outstanding stock of government debt at the end of period t. That is, $B_t^g > 0$ means that the government is a net borrower in period t, while $B_t^g < 0$ means that the government is a net lender in period t. There is a real interest rate of r_t that the government must pay on its debt.

Assuming that the government does not alter the money supply, the government's budget constraint becomes:

$$G_t + r_{t-1}B_{t-1}^g = \mathcal{T}_t + (B_t^g - B_{t-1}^g). \tag{14.1}$$

The left hand side gives expenditures of the government in period t. Notice that the government not only has to pay for its direct expenditures in period t, G_t, it must also *service the debt* by paying the interest charges $r_{t-1}B_{t-1}^g$. Of course, if the government is a net lender, then B_t^g is negative and it is collecting revenue from its holdings of other agents' debt.

The right hand side of the government budget constraint gives revenues in period t. The government raises revenue directly from the household sector by collecting taxes $\mathcal{T}_t$. In addition, it can raise revenue by *issuing net new debt* in the amount $B^g_t - B^g_{t-1}$.

Government debt is a *stock* while government deficits are a *flow.* Think of the debt as water in a bathtub: tax revenue is the water flowing out of drainhole and spending is water running in from the tap. In addition, if left to itself, the water grows (reflecting the interest rate). Each period, the level of water in the tub goes up or down (depending on $G_t, \mathcal{T}_t$ and r_t) by the amount $B^g_t - B^g_{t-1}$.

Call the *core deficit* the difference between real government purchases G_t and real government tax revenue $\mathcal{T}_t$. In the same way, define the *reported deficit* (or simply the deficit) to be the difference between all government spending, $G_t + r_{t-1}B^g_{t-1}$ and revenues from taxes $\mathcal{T}_t$. Thus:

$$\begin{aligned}
(\text{core deficit})_t &= G_t - \mathcal{T}_t, \text{ and:}\\
(\text{reported deficit})_t &= G_t - \mathcal{T}_t + r_{t-1}B^g_{t-1}.
\end{aligned}$$

The reported deficit is what is reported in the media each year as the government wrangles over the deficit. The U.S. has been running a core surplus since about 1990.

We can convert the period-by-period budget constraint in equation (14.1) into a single, infinite-horizon, budget constraint. For the rest of this chapter we will assume that the real interest rate is constant, so that $r_t = r$ all periods t. Assume further (again, purely for simplicity) that the government does not start with a stock of debt or with any net wealth, so $B^g_{-1} = 0$. Thus for convenience rewrite equation (14.1) as:

$$G_t + (1+r)B^g_{t-1} = \mathcal{T}_t + B^g_t.$$

The government's period-by-period budget constraints, starting with period zero, will therefore evolve as:

$$\begin{aligned}
(t=0) \qquad\qquad & G_0 + (1+r)\cdot 0 = \mathcal{T}_0 + B^g_0, \text{ so:}\\
& B^g_0 = G_0 - \mathcal{T}_0.\\
(t=1) \qquad\qquad & G_1 + (1+r)B^g_0 = \mathcal{T}_1 + B^g_1, \text{ so:}\\
& B^g_0 = \frac{1}{1+r}(\mathcal{T}_1 - G_1) + \frac{1}{1+r}B^g_1.\\
(t=2) \qquad\qquad & G_2 + (1+r)B^g_1 = \mathcal{T}_2 + B^g_2, \text{ so:}\\
& B^g_1 = \frac{1}{1+r}(\mathcal{T}_2 - G_2) + \frac{1}{1+r}B^g_2.\\
(t=3) \qquad\qquad & G_3 + (1+r)B^g_2 = \mathcal{T}_3 + B^g_3, \text{ so:}\\
& B^g_2 = \frac{1}{1+r}(\mathcal{T}_3 - G_3) + \frac{1}{1+r}B^g_3.
\end{aligned}$$

Now recursively substitute backwards for B^g_t in each equation. That is, for the $t = 2$ budget constraint, substitute out the B_2^g term from the $t = 3$ budget constraint to form:

$$G_2 + (1+r)B_1^g = \mathcal{T}_2 + \frac{1}{1+r}(\mathcal{T}_3 - G_3) + \frac{1}{1+r}B_3^g, \text{ so:}$$
$$B_1^g = \frac{1}{1+r}(\mathcal{T}_2 - G_2) + \left(\frac{1}{1+r}\right)^2 (\mathcal{T}_3 - G_3) + \left(\frac{1}{1+r}\right)^2 B_3^g.$$

Eventually, this boils down to:

$$B_0^g = \frac{1}{1+r}(\mathcal{T}_1 - G_1) + \left(\frac{1}{1+r}\right)^2 (\mathcal{T}_2 - G_2) + \left(\frac{1}{1+r}\right)^3 (\mathcal{T}_3 - G_3) + \left(\frac{1}{1+r}\right)^3 B_3^g.$$

Since we also know that $B_0^g = G_0 - \mathcal{T}_0$ we can rewrite this as:

$$G_0 - \mathcal{T}_0 = \frac{1}{1+r}(\mathcal{T}_1 - G_1) + \left(\frac{1}{1+r}\right)^2 (\mathcal{T}_2 - G_2) + \left(\frac{1}{1+r}\right)^3 (\mathcal{T}_3 - G_3) + \left(\frac{1}{1+r}\right)^3 B_3^g.$$

Collect all of the G_t terms on the left hand side and all of the $\mathcal{T}_t$ terms on the right hand side to produce:

$$G_0 + \frac{1}{1+r}G_1 + \left(\frac{1}{1+r}\right)^2 G_2 + \left(\frac{1}{1+r}\right)^3 G_3 =$$
$$\mathcal{T}_0 + \frac{1}{1+r}\mathcal{T}_1 + \left(\frac{1}{1+r}\right)^2 \mathcal{T}_2 + \left(\frac{1}{1+r}\right)^3 \mathcal{T}_3 + \left(\frac{1}{1+r}\right)^3 B_3^g.$$

In the same way, we can start solving backwards from any period $j \geq 0$ to write the government's budget constraint as:

$$\sum_{t=0}^{j} \left(\frac{1}{1+r}\right)^t G_t = \sum_{t=0}^{j} \left(\frac{1}{1+r}\right)^t \mathcal{T}_t + \left(\frac{1}{1+r}\right)^j B_j^g.$$

If we assume that:

$$\lim_{j\to\infty} \left(\frac{1}{1+r}\right)^j B_j^g = 0, \tag{14.2}$$

then we can continue to recursively substitute indefinitely (that is, we can let j grow arbitrarily large), to produce the single budget constraint:

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t G_t = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \mathcal{T}_t. \tag{14.3}$$

Notice that the left hand side is the present discounted value of government expenditures while the right hand side is the present discounted value of government tax revenues. The government debt terms, B_t^g, have disappeared, since, at the limit *all government borrowing* *must be repaid.* The condition in equation (14.2), sometimes known as a *transversality condition*, prohibits the government from always borrowing to pay its debt. At some point in the future, all government expenditures must be backed by government tax revenues.

In future sections, we will mainly work with constraints of the form in equation (14.3) to find optimal sequences of tax revenue $\mathcal{T}_t$ and then infer what the sequence of government debt must be.

14.2 Barro-Ricardo Equivalence

Barro-Ricardo Equivalence is the statement that the timing of government taxes do not matter, since households internalize the government budget constraint and save to pay the expected future taxes. This is an old idea, first formulated by David Ricardo in the 19th century, that has returned to prominence with the 1974 paper "Are Government Bonds Net Wealth?" by Robert Barro.[1] In that paper, Barro argued that debt-financed tax cuts could not affect output, since households would use the increased net income to save for the coming increased taxes. This argument was of particular interest during the early 1980s when debt-financed tax cuts were a centerpiece of the government's economic strategy. In this section we will examine the proposition in a simple two-period model and then again in an infinite horizon model.

Assumptions for Barro-Ricardo Equivalence

Since the time path of government debt is determined entirely by the difference between spending and taxes, Barro-Ricardo equivalence says that the optimal path of government debt is indeterminate: only the present discounted values of spending and taxes matter. Barro-Ricardo equivalence rests on three key assumptions, and we will have to break at least one of them to get a determinate theory of optimal government debt. Barro-Ricardo equivalence holds if:

1. There is a perfect capital market, on which the government and households can borrow and lend as much as desired without affecting the (constant) real interest rate.
2. Households either live forever or are altruistic towards their offspring.
3. The government can use lump-sum taxes.

Since Barro-Ricardo Equivalence requires the government and households to completely smooth out transitory spikes in spending or taxes, it is obvious why a perfect capital market is important. If households were not altruistic towards their offspring, and did not

[1] Barro, Robert J. (1974) "Are Government Bonds Net Wealth?" *Journal of Political Economy* 82(6) pp1095-1117.

live forever, then they would consume from a debt-financed tax cut without saving and bequeathing enough to their offspring to repay the debt. Finally, if the government cannot use lump-sum taxes, then large taxes cause large distortions, encouraging the government to use low taxes to spread the deadweight loss out over several periods. In the next section, we force the government to use distortionary taxes, which breaks Barro-Ricardo Equivalence.

A Two-Period Example

Consider a government which must make real expenditures of $\{G_0, G_1\}$. It levies lump-sum taxes each period of $\{\mathcal{T}_0, \mathcal{T}_1\}$. The household has a fixed endowment stream of $\{Y_0, Y_1\}$. Both the government and the household have access to a perfect bond market, and can borrow and lend any amount at the constant real interest rate r. The government's initial stock of debt, $B^g_{-1} = 0$, and the government must repay all that it borrows by the end of period $t = 1$.

The household has preferences over consumption streams $\{C_0, C_1\}$ given by:

$$U(C_0, C_1) = u(C_0) + \beta u(C_1),$$

where $0 < \beta < 1$. We assume that $u' > 0, u'' < 0$. The government's two-period (flow) budget constraints are:

$$(t = 0) \qquad G_0 = \mathcal{T}_0 + B^g_0, \text{ and:}$$

$$(t = 1) \qquad G_1 + (1 + r)B^g_0 = \mathcal{T}_1.$$

These can be collapsed (by substituting out the debt term B^g_0) into a single budget constraint, expressed in terms of present discounted value:

$$G_0 + \frac{1}{1+r}G_1 = \mathcal{T}_0 + \frac{1}{1+r}\mathcal{T}_1. \tag{14.4}$$

This is the form of the government's budget constraint with which we will work. The household's two-period (flow) budget constraints are:

$$(t = 0) \qquad C_0 + \mathcal{T}_0 + B_0 = Y_0, \text{ and:}$$

$$(t = 1) \qquad C_1 + \mathcal{T}_1 = Y_1 + (1 + r)B_0.$$

Here we are using Barro's notation that, for private individuals, B_t denotes the stock of *savings* at the end of period t. If $B_t > 0$ then the household is a net lender. Collapsing the two one-period budget constraints into a single present-value budget constraint produces:

$$C_0 + \frac{1}{1+r}C_1 = (Y_0 - \mathcal{T}_0) + \frac{1}{1+r}(Y_1 - \mathcal{T}_1). \tag{14.5}$$

Notice that the government's lump sum taxes, $\mathcal{T}_t$, which form revenue for the government, are a cost to the household.

We will now use equation (14.4) to rewrite equation (14.5) without the tax terms. Notice that equation (14.5) may be written as:

$$C_0 + \frac{1}{1+r}C_1 = Y_0 + \frac{1}{1+r}Y_1 - \left(\mathcal{T}_0 + \frac{1}{1+r}\mathcal{T}_1\right).$$

But from the government's present-value budget constraint equation (14.4) we know that:

$$\mathcal{T}_0 + \frac{1}{1+r}\mathcal{T}_1 = G_0 + \frac{1}{1+r}G_1.$$

Thus we can rewrite the household's present-value budget constraint as:

$$C_0 + \frac{1}{1+r}C_1 = Y_0 + \frac{1}{1+r}Y_1 - \left(G_0 + \frac{1}{1+r}G_1\right).$$

Notice that the household's budget constraint no longer contains tax terms $\mathcal{T}_t$. Instead, the household has internalized the government's present-value budget constraint, and uses the perfect bond market to work around any fluctuations in net income caused by sudden increases or decreases in taxes.

An Infinite-Horizon Example

The infinite horizon version is a very simple extension of the previous model. Now governments will have a known, fixed, sequence of real expenditures $\{G_t\}_{t=0}^{\infty}$ that they will have to finance with some sequence of lump-sum taxes $\{\mathcal{T}_t\}_{t=0}^{\infty}$. The household has some known endowment sequence $\{Y_t\}_{t=0}^{\infty}$. Both the household and the government can borrow and lend freely on a perfect bond market at the constant interest rate r.

The household lives forever and has preferences over sequences of consumption $\{C_t\}_{t=0}^{\infty}$ of:

$$U(\{C_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(C_t),$$

where $0 < \beta < 1$. Here we again assume that $u' > 0, u'' < 0$. To make the notation in this section simpler, define:

$$\mathbf{G} = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t G_t \qquad \mathbf{T} = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \mathcal{T}_t$$

$$\mathbf{Y} = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t Y_t \qquad \mathbf{C} = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t C_t.$$

That is, $\mathbf{G}$ is the present discounted value of government spending, $\mathbf{T}$ is the present discounted value of government revenue, $\mathbf{Y}$ is the present discounted value of the household's endowment stream and $\mathbf{C}$ the present discounted value of the household's consumption stream.

The government's present-value budget constraint, as in equation (14.3), may now be written:

$$\mathbf{G} = \mathbf{T}.$$

The household's present-value budget constraint, in the same way, may be written:

$$\mathbf{C} = \mathbf{Y} - \mathbf{T}.$$

But since the government budget constraint requires $\mathbf{T} = \mathbf{G}$, the household's budget constraint becomes:

$$\mathbf{C} = \mathbf{Y} - \mathbf{G}.$$

Once again, the timing of taxes ceases to matter. The household only cares about the present discounted value of government spending.

As a final step, we shall solve the household's problem. For simplicity, assume that $1 + r = \beta^{-1}$. The household's Lagrangian is:

$$\mathcal{L}(\{C_t\}_{t=0}^{\infty}, \lambda) = \sum_{t=0}^{\infty} \beta^t u(C_t) + \lambda(\mathbf{Y} - \mathbf{G} - \mathbf{C}).$$

To find the optimal choices of consumption given the constraint, we maximize the Lagrangian with respect to consumption. The first-order necessary conditions for maximization are formed by taking the derivative with respect to consumption in some typical period j, and from the constraint. Recall that:

$$\frac{\partial \mathbf{C}}{\partial C_j} = \left(\frac{1}{1+r}\right)^j.$$

So the first-order conditions are:

$$\beta^j u'(C_j) - \lambda \left(\frac{1}{1+r}\right)^j = 0, \text{ for all } j = 0, \ldots, \infty, \text{ and:}$$
$$\mathbf{C} = \mathbf{Y} - \mathbf{G}.$$

With the assumption that $1 + r = \beta^{-1}$, we find that:

$$u'(C_j) = \lambda, \text{ for all } j = 0, 1, \ldots, \infty.$$

But λ is constant, so $u'(C_j)$ must also be constant. We conclude that consumption is also constant, $C_j = C^\star$ in all periods j. If consumption is constant at $C^\star$, we can substitute back into the budget constraint to find $C^\star$, using the fact that $C_t = C^\star$ for all t:

$$\begin{aligned}
\mathbf{Y} - \mathbf{G} = \mathbf{C} &= \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t C_t \\
&= \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t C^\star \\
&= C^\star \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \\
&= C^\star \frac{1}{1 - \frac{1}{1+r}} \\
&= C^\star \frac{1+r}{r}, \text{ so:} \\
C^\star &= \frac{r}{1+r}(\mathbf{Y} - \mathbf{G}).
\end{aligned}$$

Thus household consumption is constant over time, increasing in $\mathbf{Y}$ and decreasing in $\mathbf{G}$. Household consumption is utterly unaffected by the timing of the taxes $\mathcal{T}_t$ used to finance the government's spending.

14.3 Preliminaries for the Ramsey Problem

Before we lay out the Ramsey model, we are going to need to define some terms. In particular, readers may be unfamiliar with *excise taxes*, which are used extensively in this chapter. Also, we will define in general terms the structure of Ramsey problems. Finally, we will define *indirect utility*, an important concept with which the reader may be unfamiliar.

Excise Taxes

An *excise tax* is a constant tariff levied on each unit of a good consumed. An example would be a \$1/gallon gasoline tax, or a \$0.25/pack cigarette tax. These are not sales taxes. Sales taxes are levied as a percentage of the total value of the goods purchased. Excise taxes are unaffected by the price of the taxed good. If there were a vector of n goods $\{x_i\}_{i=1}^n$, with an associated vector of prices $\{p_i\}_{i=1}^n$, and a consumer had m total dollars to spend on these goods, her budget constraint would be:

$$\sum_{i=1}^{n} p_i x_i \leq m.$$

Now the government levies an excise tax of τ_i on each good $i = 1 \dots n$. The consumer's budget constraint becomes:

$$\sum_{i=1}^{n} (p_i + \tau_i)\, x_i \leq m,$$

where the price paid by the consumer is now $p_i + \tau_i$. What would a sales tax look like?[2] Think of excise taxes like this: for each good x_i the consumer buys, she pays p_i to the firm, and τ_i to the government.

Under an excise tax system the government's revenue $\mathcal{H}(\cdot,\cdot)$ from the tax system, without taking into consideration the household's reactions (see Chapter 13), is:

$$\mathcal{H}(x_1, \dots, x_n; \tau_1 \dots, \tau_n) = \sum_{i=1}^{n} \tau_i x_i.$$

Households will adjust their choices of consumption x_i, $i = 1 \dots n$ in response to the taxes (this plays the role of $a_{\max}$ from Chapter 13). Thus, taking into consideration the household's best response, the government raises:

$$\mathcal{T}(\tau_1 \dots \tau_n) = \mathcal{H}(x_1^\star, \dots, x_n^\star; \tau_1, \dots, \tau_n) = \sum_{i=1}^{n} \tau_i x_i^\star(p_1, \dots, p_n; m; \tau_1, \dots, \tau_n).$$

Here $x_i^\star(\cdot;\cdot;\cdot)$ is the household's *Marshallian demand* for good i. As you recall from intermediate microeconomic theory, the Marshallian demand by an agent for a product gives the quantity of the product the agent would buy given prices and her income.

Structure of the Ramsey Problem

The government announces a sequence of excise tax rates $\{\tau_t\}_{t=0}^{\infty}$, which households take as given in making their decisions about consumption, borrowing and saving. This is actually quite a strong assumption, when you stop to think about it. The government has committed to a sequence of actions, when deviation might help it raise more revenue. What mechanism does a sovereign government have to enforce its commitment? Policies change, heads of state topple and constitutions are rewritten every year. Quite a bit of extremely interesting research centers on how governments ought to behave when they cannot credibly commit to a policy and all the agents in the model know it. See Chapter 19

[2] Okay, I'll tell you. Let's say the government levies a sales tax of t_i on each good i. Then the agent's budget constraint becomes:

$$\sum_{i=1}^{n} (1 + t_i)\, p_i x_i \leq m,$$

where now the consumer owes $t_i p_i x_i$ on each good purchased.

for a discussion of commitment in the context of a Ramsey problem in monetary policy. In that chapter we introduce the game-theoretic concepts required to model the strategic interactions of the private sector and the government.

So our benevolent government will take the purchasing behavior of its citizens (in the form a representative household) in response to its announced set of taxes $\{\tau_t\}_{t=0}^{\infty}$ as given. It will seek to raise some exogenous, known, amount of money sufficient, in present-value terms, to pay for the stream of real government expenditures on goods and services, $\{G_t\}_{t=0}^{\infty}$. These expenditures will not affect the representative household's utility or output in a meaningful way: they will be used to fight a war, or, more succinctly, thrown into the ocean. Many sequences of taxes will pay for the government's stream of purchases. Our government will choose among them by finding the tax sequence that maximizes the representative household's indirect utility.

Indirect Utility

The technical definition of indirect utility is the utility function with the choice variables replaced by their optimal values. Consider for example the following two-good problem. The utility function is:

$$U(c_1, c_2) = \ln(c_1) + \gamma \ln(c_2), \tag{14.6}$$

where $\gamma > 0$, and the budget constraint is:

$$p_1 c_1 + p_2 c_2 \leq m.$$

The Lagrangian is:

$$\mathcal{L}(c_1, c_2, \lambda) = \ln(c_1) + \gamma \ln(c_2) + \lambda(m - p_1 c_1 - p_2 c_2).$$

The first-order conditions are thus:

$$\begin{aligned} \frac{1}{c_1} - \lambda p_1 &= 0, \\ \frac{\gamma}{c_2} - \lambda p_2 &= 0, \text{ and:} \\ p_1 c_1 + p_2 c_2 &= m. \end{aligned}$$

Combined with the budget constraint, these imply that:

$$c_2 = \gamma \frac{p_1}{p_2} c_1, \text{ so:}$$

$$c_1 = \frac{1}{1+\gamma} \frac{m}{p_1}, \text{ and:} \tag{14.7}$$

$$c_2 = \frac{\gamma}{1+\gamma} \frac{m}{p_2}. \tag{14.8}$$

To find the indirect utility function, we substitute the optimal policies in equations (14.7) and (14.8) into the utility function in equation (14.6). Call the indirect utility function $V(p_1, p_2, m)$. It is how much utility the household can achieve at prices p_1, p_2 and at income m when it is optimizing. Thus, in this case:

$$\begin{aligned} V(p_1, p_2, m) &= \ln\left(\frac{1}{1+\gamma}\frac{m}{p_1}\right) + \gamma\ln\left(\frac{\gamma}{1+\gamma}\frac{m}{p_2}\right) \\ &= (1+\gamma)\ln(m) - \ln(p_1) - \ln(p_2) - (1+\gamma)\ln(1+\gamma) + \gamma\ln(\gamma). \end{aligned} \tag{14.9}$$

So we can see immediately the effect on maximized utility of an increase in wealth m, or of an increase in the prices p_1 and p_2. As we expect, optimized utility is increasing in wealth and decreasing in the prices.

Annuities

In this chapter we will often characterize income streams in terms of an *annuity.* An annuity is one of the oldest financial instruments, and also one of the simplest. In essence an annuity is a constant payment each period in perpetuity. Thus if one has an annuity of \$100, one can be assured of a payment of \$100 each year for the rest of one's life, and one may also assign it to one's heirs after death.

Risk averse agents with a known but fluctuating income stream of $\{y_t\}_{t=0}^{\infty}$ may, depending on the interest rate and their discount factor, want to convert it to an annuity, paying a constant amount a each period, of the same present discounted value. Given a constant net interest rate of r, it is easy to determine what a must be. We call a the *annuity value* of the income stream $\{y_t\}_{t=0}^{\infty}$.

We begin by calculating the present discounted value $\mathbf{Y}$ of the income stream $\{y_t\}_{t=0}^{\infty}$:

$$\mathbf{Y} \equiv \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t.$$

We know that the present discounted value of an annuity of a is just:

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t a.$$

For the present discounted value of the income stream and the annuity to be equal, a must satisfy:

$$\mathbf{Y} = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t a = \frac{a}{1-1/(1+r)} = \frac{(a)(1+r)}{r}, \text{ so:}$$

$$a = \frac{r}{1+r}\mathbf{Y}.$$

A reasonable value of r is around 0.05, which means that $r/(1+r)$ is $1/21$.

14.4 The Ramsey Optimal Tax Problem

The Household's Problem

Consider a household with a known endowment stream $\{y_t\}_{t=0}^{\infty}$. This household orders infinite sequences of consumption $\{c_t\}_{t=0}^{\infty}$ as:

$$U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t \ln(c_t), \tag{14.10}$$

where $0 < \beta < 1$. To get a nice closed form for consumption we are going to assume logarithmic preferences.

There is a perfect bond market on which the household may borrow and lend at a constant real interest rate r which we assume satisfies $1 + r = \beta^{-1}$.

The household faces a known sequence of *excise taxes* $\{\tau_t\}_{t=0}^{\infty}$ levied by the government (see the above discussion for a review of excise taxes). Since there is only one consumption good in each period, we can safely take the within-period price of the consumption good to be unity. Thus in some period t, if the household consumes c_t, expenditures must be $c_t + \tau_t c_t$ or more simply $(1 + \tau_t)c_t$. Hence the present discounted value of expenditures including the tax bill is:

$$\text{PDV}_{\text{expenditures}} = \sum_{t=0}^{\infty}(1+r)^{-t}c_t + \sum_{t=0}^{\infty}(1+r)^{-t}\tau_t c_t = \sum_{t=0}^{\infty}(1+r)^{-t}(1+\tau_t)c_t.$$

The household's present discounted value of expenditures must equal the present discounted value of the endowment stream. Hence its budget constraint is:

$$\sum_{t=0}^{\infty}(1+r)^{-t}(1+\tau_t)c_t \leq \sum_{t=0}^{\infty}(1+r)^{-t}y_t \equiv \mathbf{Y}. \tag{14.11}$$

Here I have defined the term $\mathbf{Y}$ to be the present discounted value of the income sequence $\{y_t\}_{t=0}^{\infty}$. This is merely for convenience.

Hence the household's Lagrangian is:

$$\mathcal{L}(\{c_t\}_{t=0}^{\infty}, \lambda) = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \lambda \left(\mathbf{Y} - \sum_{t=0}^{\infty}(1+r)^{-t}(1+\tau_t)c_t \right). \tag{14.12}$$

The first-order condition of equation (14.12) with respect to consumption in the arbitrary period j is:

$$\frac{\beta^j}{c_j} - \lambda(1+r)^{-j}(1+\tau_j) = 0, \text{ for all } j = 0, \dots, \infty.$$

With the assumption that $1/(1+r) = \beta$, and with a certain amount of manipulation, we can write this as:

$$c_j(1+\tau_j) = 1/\lambda, \text{ for all } j = 0\ldots\infty. \tag{14.13}$$

Notice that this last equation implies expenditures will be constant across all periods. In periods with relatively higher excise tax rates, consumption will decrease exactly enough to keep the dollar outlays exactly the same as in every other period. This is an artifact of log preferences and not a general property of this problem. However, it does simplify our job enormously.

The next step will be to substitute the optimal consumption plan in equation (14.13) into the budget constraint in equation (14.11) to determine how much the household spends each period. Substituting, we find:

$$\sum_{t=0}^{\infty}(1+r)^{-t}\frac{1}{\lambda} = \mathbf{Y}.$$

Taking out the $1/\lambda$ term (because it does not vary with t), we find that:

$$\frac{1}{\lambda} = \frac{\mathbf{Y}}{\sum_{t=0}^{\infty}(1+r)^{-t}} \equiv \mathbf{W}. \tag{14.14}$$

In other words, $1/\lambda$ is equal to the *annuity value* of the endowment stream (which I have named $\mathbf{W}$ for convenience). Of course from equation (14.13) we conclude that:

$$c_t^{\star} = \frac{\mathbf{W}}{1+\tau_t}, \text{ for all } t = 0, 1, \ldots, \infty. \tag{14.15}$$

Here I denote the optimal consumption decision as $c_t^{\star}$.

The Household's Indirect Utility

We are now ready to calculate the household's indirect utility function. Substituting the optimal policy in equation (14.15) into the preferences in equation (14.10) produces:

$$\begin{aligned}
V(\{\tau_t\}_{t=0}^{\infty}, \mathbf{W}) &= \sum_{t=0}^{\infty}\beta^t \ln(c_t^{\star}) \\
&= \sum_{t=0}^{\infty}\beta^t \ln\left(\frac{\mathbf{W}}{1+\tau_t}\right) \\
&= \sum_{t=0}^{\infty}\beta^t \left[\ln(\mathbf{W}) - \ln(1+\tau_t)\right] \\
&= -\sum_{t=0}^{\infty}\beta^t \ln(1+\tau_t) + \frac{\ln(\mathbf{W})}{1-\beta}. \qquad (14.16)
\end{aligned}$$

Here we are using some more convenient properties of the log function to simplify our result. Notice that $V(\cdot,\cdot)$ is decreasing in tax rates τ_t and increasing in the annuity value of wealth, $\mathbf{W}$.

The Government's Problem

The government faces a known unchangeable stream of period real expenditures $\{G_t\}_{t=0}^{\infty}$ and can borrow and lend at the same rate $1+r=\beta^{-1}$ as the household. Define:

$$\mathbf{G} \equiv \sum_{t=0}^{\infty}(1+r)^{-t}G_t,$$

i.e., let $\mathbf{G}$ denote the PDV of government expenditures. These government expenditures do not affect the household's utility or output in any meaningful way.

The government realizes revenue only from the excise tax it levies on the household. Hence each period the tax system produces revenues of:

$$\mathcal{H}_t(c_t;\tau_t) = \tau_t c_t.$$

But of course consumption is itself a function of taxes. The government takes as given the household's decisions. From equation (14.15) we know we can rewrite this as:

$$\mathcal{T}_t(\tau_t) = \mathcal{H}_t\left[c_t^{\star}(\tau_t);\tau_t\right] = \tau_t c_t^{\star}(\tau_t) = \mathbf{W}\frac{\tau_t}{1+\tau_t}.$$

The government's present-value budget constraint is:

$$\sum_{t=0}^{\infty}(1+r)^{-t}\mathcal{T}_t \leq \mathbf{G},\ \text{or:}$$

$$\sum_{t=0}^{\infty}(1+r)^{-t}\mathbf{W}\frac{\tau_t}{1+\tau_t} \leq \mathbf{G},$$

which we will find it convenient to rewrite as:

$$\sum_{t=0}^{\infty}(1+r)^{-t}\frac{\tau_t}{1+\tau_t} \leq \frac{\mathbf{G}}{\mathbf{W}}. \tag{14.17}$$

We divide by $\mathbf{W}$ merely to keep the algebra clean later.

The government maximizes the representative household's indirect utility subject to its present-value budget constraint by choice of sequences of excise taxes. Hence the government's Lagrangian is:

$$\mathcal{L}(\{\tau_t\}_{t=0}^{\infty},\mu) = -\sum_{t=0}^{\infty}\beta^t\ln(1+\tau_t) + \frac{\ln(\mathbf{W})}{1-\beta} + \mu\left(\sum_{t=0}^{\infty}(1+r)^{-t}\frac{\tau_t}{1+\tau_t} - \frac{\mathbf{G}}{\mathbf{W}}\right). \tag{14.18}$$

The government is choosing the sequence of tax rates $\{\tau_t\}_{t=0}^{\infty}$ which makes the household as happy as possible given that the government has to raise enough tax revenue to finance the war. Here μ is the multiplier on the government's budget constraint, the same way λ was the multiplier on the household's budget constraint previously. In some typical period j, where $j = 0, \ldots, \infty$, the first-order condition with respect to the tax rate is as follows:

$$-\frac{\beta^j}{1+\tau_j} + \mu(1+r)^{-j}\left(\frac{1}{1+\tau_j} - \frac{\tau_j}{(1+\tau_j)^2}\right) = 0, \text{ for all } j = 0, 1, \ldots, \infty.$$

Recall that we are assuming that $\beta = (1+r)^{-1}$. Hence we can manipulate this equation to produce:

$$\frac{1}{1+\tau_j} = \frac{1}{1+\tau_j}\mu\left(1 - \frac{\tau_j}{1+\tau_j}\right),$$

which reduces to:

$$\tau_j = \mu - 1.$$

This equation implies that the tax rate should not vary across periods (since μ is constant). This is one very important implication of our model: the optimal tax rate is constant. Thus we write:

$$\tau_t = \tau^\star, \text{ for all } t \geq 0.$$

Now let's find $\tau^\star$ by substituting into the government budget constraint in equation (14.17):

$$\frac{\mathbf{G}}{\mathbf{W}} = \sum_{t=0}^{\infty}(1+r)^{-t}\frac{\tau_t}{1+\tau_t} = \sum_{t=0}^{\infty}(1+r)^{-t}\frac{\tau^\star}{1+\tau^\star}.$$

We can rewrite this as:

$$\mathbf{W}\frac{\tau^\star}{1+\tau^\star} = \frac{\mathbf{G}}{\sum_{t=0}^{\infty}(1+r)^{-t}}. \tag{14.19}$$

Notice that this says that the amount of revenue collected by the government each period is constant and equal to the annuity value of government expenditures. Thus the government collects the same amount of revenue each period, running deficits when it has unusually high expenditures and surpluses when expenditures are low.

Implications for the Path of Debt.

Imagine a government that has to fight a war in period 0, and makes no other expenditures in all other periods. Let the cost of a war be unity. Thus government expenditures satisfy:

$$G_t = \begin{cases} 1, & t = 0 \\ 0, & t \geq 1. \end{cases}$$

Hence the present discounted value of government expenditures is:

$$\mathbf{G} = \sum_{t=0}^{\infty}(1+r)^{-t}G_t = (1+r)^0(1) + \sum_{t=1}^{\infty}(1+r)^{-t}(0) = 1.$$

We know from equation (14.19) that the optimal tax rate $\tau^\star$ satisfies:

$$\mathbf{W}\frac{\tau^\star}{1+\tau^\star} = \frac{1}{\sum_{t=0}^{\infty}(1+r)^{-t}} = 1 - \frac{1}{1+r} = \frac{r}{1+r}, \text{ so:}$$
$$\frac{\tau^\star}{1+\tau^\star} = \frac{r}{1+r}\frac{1}{\mathbf{W}}.$$

For the sake of argument, say that the household has a constant endowment $y_t = 1$ all $t \geq 0$. In other words, the government has to fight a war in the first period that costs as much as the total economy-wide wealth in that period. If this is the case then we can find $\mathbf{Y}$ and $\mathbf{W}$:

$$\mathbf{Y} = 1 \cdot \sum_{t=0}^{\infty}(1+r)^{-t} = \frac{1}{1-\frac{1}{1+r}} = \frac{1+r}{r}.$$
$$\mathbf{W} = \frac{\mathbf{Y}}{\sum_{t=0}^{\infty}(1+r)^{-t}} = \mathbf{Y}\left(\frac{1+r}{1+r-1}\right)^{-1} = 1.$$

This makes sense: the annuity value of $\mathbf{Y}$ is just the infinite flow of constant payments that equals $\mathbf{Y}$. But since $\mathbf{Y}$ is made up of the infinite flow of constant payments of $y_t = 1$ each period, then the annuity value must also be unity. Now we can find $\tau^\star$ from:

$$\frac{\tau^\star}{1+\tau^\star} = \frac{r}{1+r},$$

which means that $\tau^\star = r$. In other words, the optimal tax rate $\tau^\star$ is simply the interest rate r. Government tax revenues each period are:

$$\mathcal{T}_t = \frac{r}{1+r}, \text{ for all } t \geq 0.$$

The government is collecting this relatively small amount each period in our example.

Notice what this implies for the path of deficits (and hence debt). In period $t = 0$ the government collects $r/(1+r)$ and pays out 1 to fight its war. Hence the core deficit in period $t = 0$ is:

$$G_0 - \mathcal{T}_0 = 1 - \frac{r}{1+r} = \frac{1}{1+r}.$$

In all subsequent periods, the government spends nothing and collects the usual amount, hence running core surpluses (or negative core deficits) of:

$$G_t - \mathcal{T}_t = 0 - \frac{r}{1+r} = -\frac{r}{1+r}, \text{ for all } t = 1, 2, \dots, \infty.$$

From the government's flow budget constraint we know that:

$$G_t + (1+r)B^g_{t-1} = \mathcal{T}_t + B^g_t, \text{ for all } t \geq 0.$$

Hence from period 1 onward, while the government is repaying its debt from period 0:

$$B^g_t = -\frac{r}{1+r} + (1+r)B^g_{t-1}, \text{ for all } t \geq 1.$$

From this is it easy to see that the government debt, after the war, is constant at:

$$B^g_t = \frac{1}{1+r}, \text{ for all } t = 1, 2, \ldots, \infty.$$

Each period, the government raises just enough revenue to pay the interest cost on this debt and roll it over for another period. Does this violate our transversality condition in equation (14.2)? No, because the government debt is not exploding, it is merely constant. Thus, from equation (14.2):

$$\lim_{t\to\infty} \left(\frac{1}{1+r}\right)^t B^g_t = \lim_{t\to\infty} \left(\frac{1}{1+r}\right)^t \left(\frac{1}{1+r}\right) = \lim_{t\to\infty} \left(\frac{1}{1+r}\right)^{t+1} = 0.$$

So the transversality condition is satisfied by the government's optimal debt plan.

Exercises

Exercise 14.1 (Easy)
Supply the following facts. Most can be found in the Barro textbook.

1. What was the ratio of nominal outstanding public debt to GNP for the U.S. in 1996?
2. About what was the highest debt/GNP ratio experienced by the U.S. since 1900? In what year?
3. According to Barro, about what was the highest marginal tax rate paid by the "average" American since 1900? In what year?
4. Illinois has a standard deduction of about $2000. Every dollar of income after that is taxed at a constant rate (a "flat" tax). What is that rate?

Exercise 14.2 (Moderate)
The government must raise a sum of G from the representative household using only excise taxes on the two goods in the economy. The household has preferences over the two goods of:

$$U(x_1, x_2) = \ln(x_1) + x_2.$$

Variable	Definition
B_t	Real end-of-period savings by the household sector.
B_t^g	Real par value of outstanding end-of-period government debt.
B_{-1}^g	Initial stock of government debt, $B_{-1}^g = 0$.
C_t, c_t	Aggregate (household) consumption at t.
G_t	Real government spending at t.
Y_t, y_t	Aggregate (household) income at t.
r_t	Real net return on end-of-period debt.
$\mathbf{C}$	Present discounted value of consumption stream, $\{C_t\}_{t=0}^{\infty}$.
$\mathbf{G}$	Present discounted value of government spending stream, $\{G_t\}_{t=0}^{\infty}$.
$\mathbf{T}$	Present discounted value of tax revenue stream, $\{\mathcal{T}_t\}_{t=0}^{\infty}$.
$\mathbf{W}$	Annuity value of income stream, $\{Y_t\}_{t=0}^{\infty}$.
$\mathbf{Y}$	Present discounted value of income stream, $\{Y_t\}_{t=0}^{\infty}$.
$\mathcal{H}(a;\psi)$	Government tax policy mapping household action a and a vector of tax parameters ψ into a tax bill (see Chapter 13).
$\mathcal{T}(\psi)$	Government revenue as a function of tax policy parameters ψ when the household chooses its best response, $a_{\max}$ (see Chapter 13).
$U(\{C_t\}_{t=0}^{\infty})$	Utility from a consumption stream.
$u(C_t)$	One-period utility from consumption of C_t in period t.
$V(\{\tau_t\}_{t=0}^{\infty})$	Household's indirect utility given a stream of excise taxes $\{\tau_t\}_{t=0}^{\infty}$ and an income stream $\{Y_t\}_{t=0}^{\infty}$ with annuity value $\mathbf{W}$.

Table 14.1: Notation for Chapter 14. Note that, with the assumption that $Y_t = N_t$, variables denoted as per-capita are also expressed as fractions of GDP.

The household has total wealth of M to divide between expenditures on the two goods. The two goods have prices p_1 and p_2, and the government levies excise taxes of t_1 and t_2. The government purchases are thrown into the sea, and do not affect the household's utility or decisions. Determine the government revenue function $\mathcal{T}(t_1, t_2; p_1, p_2, M)$. Determine the household's indirect utility function in the presence of excise taxes, $V(p_1+t_1, p_2+t_2, M)$.

Hence write down the problem to determine the distortion-minimizing set of excise taxes t_1 and t_2. *You do not have to find the optimal tax rates.*

Exercise 14.3 (Easy)
Assume that a household lives for two periods and has endowments in each period of $\{y_1 = 1, y_2 = 1 + r\}$. The household can save from period $t = 1$ to period $t = 2$ on a bond market at the constant net interest rate r. If the household wishes to borrow to finance consumption in period $t = 1$ and repay in period $t = 2$, it must pay a higher net interest rate of $r' > r$. The household has preferences over consumption in period t, c_t, of:

$$V(\{c_1, c_2\}) = u(c_1) + u(c_2).$$

Assume that $u' > 0$ and $u'' < 0$.

There is a government which levies lump sum taxes of $\mathcal{T}_t$ on the household in each period, $t = 1, 2$. The government can borrow and lend at the same interest rate, r. That is, the government does not have to pay a premium interest rate to borrow. The government must raise $\mathbf{G} = 1$ in present value from the household. That is:

$$\mathcal{T}_1 + \frac{\mathcal{T}_2}{1+r} = \mathbf{G} = 1.$$

Answer the following questions:

1. Find the household's consumption in each period if it does not borrow or lend.
2. Assume that $\mathcal{T}_1 > 0$ and $\mathcal{T}_2 > 0$. Draw a set of axes. Put consumption in period $t = 2$, c_2, on the vertical axis and consumption in period $t = 1$ on the horizontal axis. Draw the household's budget set.
3. Now assume that $\mathcal{T}_1 = 0$ and $\mathcal{T}_2 = (1 + r)\mathbf{G}$. Draw another set of axes and repeat the previous exercise.
4. Now assume that $\mathcal{T}_1 = \mathbf{G}$ and $\mathcal{T}_2 = 0$. Draw another set of axes and repeat the previous exercise.
5. What tax sequence would a benevolent government choose? Why?

Exercise 14.4 (Easy)
Assume that the government can only finance deficits with debt (it cannot print money, as in the chapter). Assume further that there is a constant real interest rate on government debt of r, and that the government faces a known sequence of real expenditures $\{G_t\}_{t=0}^{\infty}$. The government chooses a sequence of taxes that produces revenues of $\{\mathcal{T}_t\}_{t=0}^{\infty}$ such that:

$$\begin{aligned}
&\mathcal{T}_0 = G_0 - 1, \text{ and:}\\
&\mathcal{T}_t = G_t, \text{ for all } t = 1, 2, \ldots, \infty.
\end{aligned}$$

Find the path of government debt implied by this fiscal policy. Does it satisfy the transversality condition? Why or why not?