Solutions to Exercises

Exercise 1.1
You may have noticed that this question glosses over the compounding issue. You were intended to assume that the APR was quoted as a simple interest rate. Accordingly, the daily interest rate is just:

$$R = \frac{16.8\%}{365} \approx 0.046027\%.$$

Exercise 1.2
This exercise glossed over the compounding issue again. Assuming no compounding over the week, the interest rate is:

$$R = \left(\frac{25}{1,000}\right)(52) = 1.3 = 130\%.$$

Exercise 1.3
The key to this question is that the units you use to measure time in the exponent are the same units of time for the resulting interest rate. For example, if you measure n in years, then solving for R gives you an annual interest rate. If you measure n in "quarters", then R will be a quarterly interest rate.

Since this question asks you to annualize the answer, you want to measure n in years. The time interval is 3 months, which is 1/4 of a year. Accordingly:

$$157.8 = \left[e^{(R)(1/4)}\right](156.7), \text{ so:}$$
$$R = 4[\ln(157.8) - \ln(156.7)] \approx 0.02798104 = 2.798104\%.$$

Exercise 1.4
You do *not* want to annualize these interest rates, so you measure n in quarters, i.e., $n = 1$:

1st quarter: $R_1 = \ln(156.7) - \ln(155.7) \approx 0.00640207 = 0.640207\%$.
2nd quarter: $R_2 = \ln(157.8) - \ln(156.7) \approx 0.00699526 = 0.699526\%$.
3rd quarter: $R_3 = \ln(158.6) - \ln(157.8) \approx 0.00505690 = 0.505690\%$.
4th quarter: $R_4 = \ln(160.0) - \ln(158.6) \approx 0.00878851 = 0.878851\%$.

You can see that by adding these four lines together, all but two terms cancel, leaving: $R = \ln(160.0) - \ln(155.7) \approx 0.02724274 = 2.724274\%$. And of course, this is precisely the formula for the annual growth using a continuous interest rate.

Exercise 1.5

$$(2)(\text{GDP}) = \left[e^{(0.02)(n)}\right](\text{GDP}), \text{ so:}$$
$$n = \frac{\ln(2)}{0.02} = 34.66 \text{ years.}$$

The Rule of 72 says that it should take about $72/2 = 36$ years, which is pretty close (it is off by about 3.9%). Of course, you are smart enough to look at:

$$n = \frac{\ln(2)}{0.02}$$

and notice that a better rule would be the "Rule of 69", but nobody is very good at dividing into 69 in their head.

Exercise 1.6
The first thing you need to do is calculate the number of whole (i.e., undivided) years this investment will require. There is some number n of years such that:

$$(1 + 0.065)^n(\$10,000) < \$15,000, \quad \text{but where:}$$
$$(1 + 0.065)^{n+1}(\$10,000) > \$15,000.$$

This implies that $n = 6$. After the 6th year, the investment has grown to:

$$(1 + 0.065)^6(\$10,000) = \$14,591.42.$$

That becomes the principal of the investment in the 7th year, since interest was able to compound at the end of the 6th year. Now you need to figure out the number of days of simple interest the investment will need in the 7th year. It is short of $15,000 by $408.58, and each day the investment earns:

$$\left(\frac{0.065}{365}\right)(\$14,591.42) = \$2.60.$$

You use these facts to calculate the required number of days:

$$\frac{\$408.58}{\$2.60} = 157.23.$$

Since interest only accrues after a full day, the investment would not earn the interest from the last 0.23 days until 158 days had passed. All in all then, the investment would require 6 years and 158 days.

Exercise 1.7
Let F_t be the acreage of forest in year t. Then:

$$F_n = F_0(1 - 0.046)^n.$$

You are looking for the n such that $F_n = (0.5)F_0$. Plugging this into the equation and taking logs of both sides yields: $n \approx 14.72$, so half will be cleared in 15 years.

Exercise 1.8

1. The relevant formula is:

$$679e^{100R} = 954.$$

so $R \approx 3.4 \times 10^{-3}$.

2. Letting x be the number of years, the relevant formula is:

$$2e^{xR} = 679{,}000{,}000.$$

You use R from the previous part. Solving yields $x \approx 5777$, so they would have left the Garden of Eden in about 4075 BCE. (Population growth was probably slower in the past, so this is likely not early enough.)

Exercise 1.9

1. You are solving for n in the following equation:

$$\$10{,}600(1.047)^n = \$15{,}400(1.017)^n.$$

This implies that $n \approx 12.85$ years, so the incomes would be the same sometime in 1996. (Nb: They were not. Japan's income was still less at that date.)

2. Just plug in the value of n:

$$\$15{,}400(1.017)^{12.85} \approx \$19{,}124.$$

Exercise 2.1

$$\max_{c,l,n_s} \{\ln(c) + \ln(l)\}, \text{ such that:}$$
$$c = 4n_s^{0.5} + (24 - l - n_s)w.$$

Exercise 2.2
First, write out the maximization problem:

$$\max_{c,l} \left\{c^{\gamma}(1-l)^{1-\gamma}\right\}, \text{ such that:}$$
$$c = y = Al^{\alpha}.$$

Plug the constraint into the objective to get an unconstrained maximization problem:

$$\max_l \left\{(Al^\alpha)^\gamma(1-l)^{1-\gamma}\right\}.$$

There is only one first-order condition:

$$\text{(FOC } l\text{)} \qquad \gamma[A(l^\star)^\alpha]^{\gamma-1}[A\alpha(l^\star)^{\alpha-1}](1-l)^{1-\gamma} = [A(l^\star)^\alpha]^\gamma(1-\gamma)(1-l^\star)^{-\gamma}.$$

After a bunch of canceling and rearranging, this reduces to:

$$l^\star = \frac{\gamma\alpha}{1-\gamma+\gamma\alpha}.$$

Plugging this back into the $c = y = f(l)$ constraint yields:

$$c^\star = A\left(\frac{\alpha\gamma}{1-\gamma+\gamma\alpha}\right)^\alpha.$$

Exercise 3.1

1. The marginal period utility is:

$$u'(c_t) = \frac{1}{2}c_t^{-\frac{1}{2}}.$$

   Plugging this into equation (3.7) yields:

$$\frac{\frac{1}{2}(c_1^\star)^{-\frac{1}{2}}}{\frac{1}{2}(c_2^\star)^{-\frac{1}{2}}} = \beta(1+R), \text{ or:}$$

$$\left(\frac{c_2^\star}{c_1^\star}\right)^{\frac{1}{2}} = \beta(1+R).$$

2. We have three unknowns: $c_1^\star$, $c_2^\star$, and $b_1^\star$. The three equations relating them are: the Euler equation above and the two budget equations. Solving these is an unpleasant exercise in algebra. Solve the Euler equation for $c_2^\star$:

$$c_2^\star = \beta^2(1+R)^2c_1^\star.$$

   Use this to remove $c_2^\star$ from the second-period budget:

$$Py_2 + b_1^\star(1+R) = P\left[\beta^2(1+R)^2c_1^\star\right].$$

   Solve this for $c_1^\star$, and plug the result into the first-period budget:

$$Py_1 = P\left[\frac{Py_2 + b_1^\star(1+R)}{P\beta^2(1+R)^2}\right] + b_1^\star.$$

This looks awful, but it reduces to:

$$b_1^\star = Py_1 - \frac{P[y_2 + y_1(1+R)]}{[1+\beta^2(1+R)](1+R)},$$

which is the answer for the household's choice of b_1. Plugging this back into the first-period budget gives the optimal c_1:

$$c_1^\star = \frac{y_2 + y_1(1+R)}{[1+\beta^2(1+R)](1+R)}.$$

Finally, we plug the answer for $c_1^\star$ into the second-period budget equation to get:

$$c_2^\star = \frac{\beta^2(1+R)[y_2 + y_1(1+R)]}{1+\beta^2(1+R)}.$$

3. In equilibrium, $b_1^\star = 0$, so:

$$Py_1 = \frac{P[y_2 + y_1(1+R^\star)]}{[1+\beta^2(1+R^\star)](1+R^\star)},$$

Solving for $R^\star$ yields:

$$R^\star = \left(\frac{y_2}{\beta^2 y_1}\right)^{\frac{1}{2}} - 1.$$

4. From the above equation, we see that an equal percentage increase in y_1 and y_2 will have no effect on the equilibrium interest rate $R^\star$, just like under logarithmic preferences.

Exercise 3.2

1.

$$\frac{\partial R^\star}{\partial \beta} = -\frac{y_2}{\beta^2 y_1} < 0.$$

Greater impatience means β decreases (say, from 0.95 to 0.9), and $R^\star$ moves in the opposite direction, so the equilibrium interest rate increases.

2.

$$\frac{\partial R^\star}{\partial y_1} = -\frac{y_2}{\beta y_1^2} < 0,$$

so smaller first-period income causes the equilibrium interest rate to increase.

Exercise 3.3

1.

$$\max_{c_1, c_2, s} \left\{\ln(c_1) + \beta \ln(c_2)\right\}, \quad \text{subject to:}$$

$$c_1 + s = e_1, \quad \text{and:}$$
$$c_2 = e_2 + (1-\delta)s.$$

2. The Lagrangean for this problem is:
$$\mathcal{L} = \ln(c_1) + \beta \ln(c_2) + \lambda_1[e_1 - c_1 - s] + \lambda_2[e_2 + (1-\delta)s - c_2].$$
The first-order conditions are:
$$\text{(FOC } c_1\text{)} \qquad \frac{1}{c_1} - \lambda_1 = 0,$$
$$\text{(FOC } c_2\text{)} \qquad \frac{\beta}{c_2} - \lambda_2 = 0, \quad \text{and:}$$
$$\text{(FOC } s\text{)} \qquad -\lambda_1 + \lambda_2(1-\delta) = 0.$$
We also have two first-order conditions for the Lagrange multipliers, but we leave those off, since they just reproduce the constraints. We can quickly solve the above equations to remove the Lagrange multipliers, giving us:
$$\frac{c_2}{c_1} = \beta(1-\delta).$$
We combine this with our two constraints to get:
$$s = \frac{\beta(1-\delta)e_1 - e_2}{(1-\delta)(1+\beta)},$$
$$c_1 = e_1 - \frac{\beta(1-\delta)e_1 - e_2}{(1-\delta)(1+\beta)}, \quad \text{and:}$$
$$c_2 = e_2 + \frac{\beta(1-\delta)e_1 - e_2}{1+\beta}.$$

3. We just take the derivatives of the above answers with respect to δ:
$$\frac{\partial c_1}{\partial \delta} = -\frac{e_2(1+\beta)}{(1-\delta)^2(1+\beta)^2} < 0,$$
$$\frac{\partial c_1}{\partial \delta} = -\frac{\beta e_1}{1+\beta} < 0, \quad \text{and:}$$
$$\frac{\partial s}{\partial \delta} = -\frac{e_2(1+\beta)}{(1-\delta)^2(1+\beta)^2} < 0.$$
Here, Maxine has learned how to defend against rats, so we are interested in δ going down. The negative derivatives above imply that all the choices change in the opposite direction, so consumption in both periods and first-period saving all increase.

Exercise 3.4

$$\max_{s_0,\dots,s_4} \left\{ \sum_{t=0}^{4} \beta^t \ln(c_t) \right\}, \text{ such that:}$$
$$c_t = (1-s_t)x_t, \text{ for } t = 0,\dots,4,$$
$$x_{t+1} = (1+\alpha)sx_t, \text{ for } t = 0,\dots,4, \text{ and:}$$
$$x_0 \text{ is some given constant.}$$

Exercise 4.1

1. Take the derivative of real money demand with respect to the interest rate R:

$$\frac{\partial \phi(R, c, \gamma/P)}{\partial R} = \left(\frac{1}{2}\right) c \left(\frac{2\gamma}{PRc}\right)^{-\frac{1}{2}} \left(-\frac{2\gamma}{PR^2c}\right) < 0,$$

   so the interest rate and real money holdings move in opposite directions. An increase in the interest rate causes the consumer to hold less real money.

2. Differentiate with respect to c:

$$\begin{aligned}\frac{\partial \phi(R, c, \gamma/P)}{\partial c} &= \left(\frac{1}{2}\right) c \left(\frac{1}{2}\right) \left(\frac{2\gamma}{PRc}\right)^{-\frac{1}{2}} \left(-\frac{2\gamma}{PRc^2}\right) + \left(\frac{2\gamma}{PRc}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right) \\ &= \left(\frac{1}{2}\right) \left(\frac{2\gamma}{PRc}\right)^{\frac{1}{2}} - \left(\frac{1}{4}\right) c \left(\frac{2\gamma}{PRc}\right)^{\frac{1}{2}} \left(\frac{1}{c}\right) \\ &= \left(\frac{1}{4}\right) \left(\frac{2\gamma}{PRc}\right)^{\frac{1}{2}} > 0,\end{aligned}$$

   so consumption and real money holdings move in the same direction. If the consumer consumes more, then the consumer will hold more real money.

3. First, we replace γ/P with α, giving us:

$$\phi(R, c, \alpha) = \left(\frac{1}{2}\right) c \left(\frac{2\alpha}{Rc}\right)^{\frac{1}{2}}.$$

   Taking the derivative with respect to α gives us:

$$\frac{\partial \phi(R, c, \gamma/P)}{\partial \alpha} = \left(\frac{1}{2}\right) c \left(\frac{1}{2}\right) \left(\frac{2\alpha}{Rc}\right)^{-\frac{1}{2}} \left(\frac{2}{Rc}\right) > 0,$$

   so real money holdings and real transactions costs move in the same direction. If the consumer faces higher real transactions costs, the consumer will hold more real money.

Exercise 5.1

The budget constraint for the first period was given by:

$$Py_1 = Pc_1 + b_1. \tag{3.2}$$

The condition for clearing the goods market in the first period was:

$$Ny_1 = Nc_2. \tag{3.10}$$

This implies $y_1 = c_1$. Plugging this into (3.2) gives:

$$\begin{gathered} Pc_1 = Pc_1 + b_1, \text{ or:} \\ b_1 = 0, \end{gathered}$$

which is the market-clearing constraint for bonds.

Exercise 5.2

The price for a good can only be zero if all consumers are satiated with that good, that is, if they cannot increase their utility by consuming more of it. In our model this is ruled out because all utility functions are strictly increasing in all arguments. This implies that the consumers always prefer to consume more of each good. If the price for a good were zero, they would demand infinite amounts, which would violate market-clearing. Therefore, with strictly increasing utility functions, all prices are positive. If utility is not strictly increasing, zero prices are possible. In that case, Walras' Law might not hold, because total demand by consumers can be less than the total endowment. The proof of Walras' Law fails once we use the fact that the price of each good is positive. On the other hand, the First Welfare Theorem still goes through, since it does not rest on the assumption of positive prices.

Exercise 6.1

1. The first-order condition with respect to l_d^a is:

   (FOC l_d^a) $$(0.5)(l_d^{a\star})^{-0.5} - w = 0.$$

   Solving for $l_d^{a\star}$ yields: $l_d^{a\star} = \frac{1}{4w^2}$. The farm's profit is:

   $$(l_d^{a\star})^{0.5} - wl_d^{a\star} = \left(\frac{1}{4w^2}\right)^{0.5} - \frac{w}{4w^2} = \frac{1}{4w} = \pi^{a\star}.$$

2. The first-order condition with respect to l_d^b is:

   (FOC l_d^b) $$(0.5)(2)(l_d^{b\star})^{-0.5} - w = 0.$$

   Solving for $l_d^{b\star}$ yields: $l_b^{d\star} = \frac{1}{w^2}$. The farm's profit is:

   $$2(l_b^{d\star})^{0.5} - wl_b^{d\star} = 2\left(\frac{1}{w^2}\right)^{0.5} - \frac{w}{w^2} = \frac{1}{w} = \pi^{b\star}.$$

3. For economy, we will work this out for an unspecified $\pi^{j\star}$, where j is either a or b. We'll plug those in later. Substitute the constraint into to objective in order to eliminate c^j. This gives us:

   $$\max_{l_s^j} \left\{ \ln(wl_s^j + \pi^{j\star}) + \ln(24 - l_s^j) \right\}.$$

   We carry out the maximization:

   (FOC l_s^j) $$\frac{w}{wl_s^j + \pi^{j\star}} + \frac{-1}{24 - l^{j\star}} = 0, \text{ so:}$$
   $$(24 - l_s^{j\star})w = wl_s^{j\star} + \pi^{j\star}.$$

Solving for $l_s^{j\star}$ yields:

$$l_s^{j\star} = \frac{24w - \pi^{j\star}}{2w}. \tag{S.8}$$

For this exercise, we are using $\pi^{a\star}$, so we plug that in to get:

$$l_s^{a\star} = \frac{24w - \frac{1}{4w}}{2w} = 12 - \frac{1}{8w^2}.$$

4. We can just re-use equation (S.8), but this time we plug in $\pi^{b\star}$, yielding:

$$l_s^{b\star} = \frac{24w - \frac{1}{w}}{2w} = 12 - \frac{1}{2w^2}.$$

5.

$$l_d^\star = 400 l_d^{a\star} + 700 l_d^{b\star} = \frac{400}{4w^2} + \frac{700}{w^2} = \frac{800}{w^2}.$$

6.

$$\begin{aligned} l_s^\star &= 400 l_s^{a\star} + 700 l_s^{b\star} \\ &= (400)\left(12 - \frac{1}{8w^2}\right) + (700)\left(12 - \frac{1}{2w^2}\right) \\ &= (1,100)(12) - \frac{400}{w^2}. \end{aligned}$$

7. We want to set $l_d^\star = l_s^\star$, or:

$$\frac{800}{w^2} = (1,100)(12) - \frac{400}{w^2},$$

which reduces to $w^2 = 1/11$, or $w \approx 0.3015$.

Exercise 6.2

1. We take the first-order condition of equation (6.9) with respect to l_d:

$$\frac{\partial \pi}{\partial l_d} = \left(\frac{7}{10}\right) A k^{\frac{3}{10}} (l_d^\star)^{-\frac{3}{10}} - w = 0 \tag{FOC l_d}$$

When we solve that for $l_d^\star$, we get:

$$l_d^\star = \left(\frac{7A}{10w}\right)^{\frac{10}{3}} k$$

2. The result is as follows:

$$\begin{aligned}
\pi^\star &= Ak^{\frac{3}{10}}(l_d^\star)^{\frac{7}{10}} - wl_d^\star \\
&= Ak^{\frac{3}{10}}\left[\left(\frac{7A}{10w}\right)^{\frac{10}{3}} k\right]^{\frac{7}{10}} - w\left(\frac{7A}{10w}\right)^{\frac{10}{3}} k \\
&= Ak^{\frac{3}{10}}\left(\frac{7A}{10w}\right)^{\frac{7}{3}} k^{\frac{7}{10}} - w\left(\frac{7A}{10w}\right)^{\frac{10}{3}} k \\
&= k\left(\frac{7A}{10w}\right)^{\frac{7}{3}}\left[A - w\left(\frac{7A}{10w}\right)\right] \\
&= \left(\frac{3A}{10}\right)\left(\frac{7A}{10w}\right)^{\frac{7}{3}} k.
\end{aligned}$$

3.

$$\max_{c,l_s}\{c^{\frac{1}{2}}(1-l_s)^{\frac{1}{2}}\}, \text{ subject to:}$$

$$c = \pi^\star + wl_s.$$

4. To begin, we can leave the $\pi^\star$ term in the Lagrangean:

$$\mathcal{L} = c^{\frac{1}{2}}(1-l_s)^{\frac{1}{2}} + \lambda[\pi^\star + wl_s - c].$$

Our first-order conditions are:

(FOC c) $$\left(\frac{1}{2}\right)(c^\star)^{-\frac{1}{2}}(1-l_s^\star)^{\frac{1}{2}} + \lambda^\star[-1] = 0, \text{ and:}$$

(FOC l_s) $$(c^\star)^{\frac{1}{2}}\left(\frac{1}{2}\right)(1-l_s^\star)^{-\frac{1}{2}}(-1) + \lambda^\star[w] = 0.$$

We leave off the FOC for λ. Combining the above FOCs to get rid of $\lambda^\star$ yields:

(S.9) $$c^\star = w(1-l_s^\star).$$

We plug this result and our expression for $\pi^\star$ into the budget equation $\pi + wl_s = c$, yielding:

$$\left(\frac{3A}{10}\right)\left(\frac{7A}{10w}\right)^{\frac{7}{3}} k = w(1-l_s^\star) - wl_s^\star.$$

Solving this for $l_s^\star$ gives us:

$$l_s^\star = \frac{1}{2}\left[1 - \frac{3}{7}\left(\frac{7A}{10w}\right)^{\frac{10}{3}} k\right].$$

When we plug this value of $l_s^\star$ back into equation (S.9), we get the optimal consumption $c^\star$:

$$c^\star = \frac{w}{2}\left[1 + \frac{3}{7}\left(\frac{7A}{10w}\right)^{\frac{10}{3}} k\right].$$

5. We just set $l_s^\star = l_d^\star$, solve for w, and call the result $w^\star$:

$$\left(\frac{7A}{10w^\star}\right)^{\frac{10}{3}} k = \frac{1}{2}\left[1 - \frac{3}{7}\left(\frac{7A}{10w^\star}\right)^{\frac{10}{3}} k\right].$$

   After a bunch of algebra, we get:

$$w^\star = \left(\frac{7A}{10}\right)\left(\frac{17k}{7}^{\frac{3}{10}}\right).$$

6. We are interested in the derivative if $w^\star$ with respect to k:

$$\frac{\partial w^\star}{\partial k} = \left(\frac{7A}{10}\right)\left(\frac{3}{10}\right)\left(\frac{17k}{7}\right)^{-\frac{7}{10}}\left(\frac{17}{7}\right) = \left(\frac{51A}{100}\right)\left(\frac{17k}{7}\right)^{-\frac{7}{10}}.$$

   Since this derivative is positive, $w^\star$ increases as k does.

7. The U.S. has a much larger stock of capital (per capita) than Mexico does. According to this model, that difference alone causes wages to be higher in the U.S. From the equation for the equilibrium wage $w^\star$, we see that increasing the per-capita capital stock k by a factor of two causes the wage to increase, but by less than a factor of two. Hence, wages between the two countries differ by less (in percentage terms) than their per-capita capital stocks.

   Of course, owners of capital try to export it to wherever labor is cheapest. In this case, the households in the U.S. try to send some of their capital to Mexico in order to take advantage of lower wages there. If this movement of capital is restricted, then the wage difference will persist, and there will be an incentive for workers to move to the country with more capital. In this case, Mexican workers will see higher wages across the border and will immigrate to the U.S. where they will earn more.

Exercise 8.1
According to the quantity theory, the inflation rate is approximately equal to the difference between the growth rate of money supply and the growth rate of output. Since the question assumes that velocity is constant, the quantity theory applies. The annual rate of inflation is therefore two percent.

Exercise 8.2
In Chapter 8 we determined that velocity is inversely related to the time spent between two trips to the bank. In Chapter 4 we saw that the time between two trips to the bank

decreases when the nominal interest rate increases. Therefore velocity and the nominal interest rate are positively related. In Section 8.3 we found out that inflation and nominal interest rates are positively related. Therefore, a high inflation rate results in high nominal interest rates and high velocity. This is also true in the real world: velocity is much higher in countries with high inflation than in countries with moderate inflation. Intuitively, high inflation means that money quickly loses value. It is therefore not attractive to hold a lot of money, so money circulates quickly. In countries with hyperinflation, wages are often paid daily, and workers usually spend wages the same day they receive them.

Exercise 9.1
Of course, the solution depends on the country you pick. As an example, Figure S.3 displays GDP and its trend for Germany. You can see that the trend does not look that much smoother than the actual series. This shows that our method of computing the trend is not especially good.

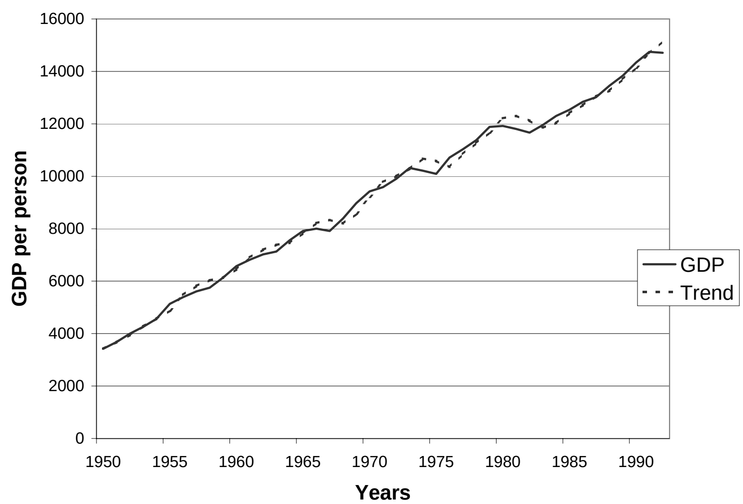


Figure S.3: GDP and Trend

Exercise 9.2
Figure S.4 shows the cyclical component for Germany. Your business cycle should look similar, unless your country is a former member of the communist block. Those countries either had radically different business cycles, or, more likely, they adjusted their statistics in order to get nice, smooth figures.

Exercise 9.3
For Germany, there are ten peaks in the cyclical component. The duration of a full cycle is between three and six years, with the average slightly above four years. The overall

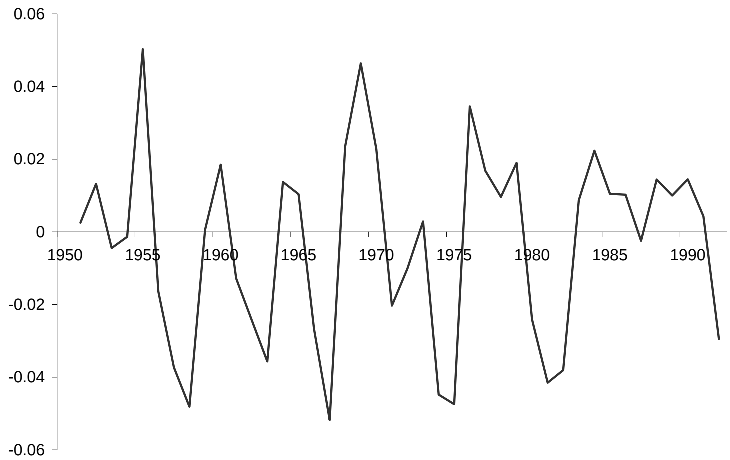


Figure S.4: The Cyclical Component

amplitude of the cycles is relatively stable. Although there are some general similarities, the cycles are of quite different shape. The process generating the cycles seems not to have changed much, however. The cycles in the fifties and sixties are not much different from those in the eighties and nineties.

Exercise 9.4
By using the resource constraints, we can write the problem as:

$$\max \ln(\sqrt{Bk_t} + \epsilon_t - i_t) + A\ln((1-\delta)k_t + i_t).$$

The first-order condition is:

$$0 = -\frac{1}{\sqrt{Bk_t} + \epsilon_t - i_t} + \frac{A}{(1-\delta)k_t + i_t}.$$

Solving for i_t, we get:

$$i_t = \frac{A[\sqrt{Bk_t} + \epsilon_t] - (1-\delta)k_t}{1+A}.$$

Using the resource constraint for the first period, we can solve for c_t:

$$c_t = \frac{\sqrt{Bk_t} + \epsilon_t + (1-\delta)k_t}{1+A}.$$

Exercise 9.5
The derivatives are:

$$\frac{\partial i_t}{\partial \epsilon_t} = \frac{A}{1+A} = 0.8, \text{ and:}$$
$$\frac{\partial c_t}{\partial \epsilon_t} = \frac{1}{1+A} = 0.2.$$

The numbers correspond to the value $A = 4$ that is used for the simulations. Investment reacts much stronger to shocks than consumption does, just as we observe in real-world data.

Exercise 9.6
Figure S.5 shows consumption and investment, and Figure S.6 is GDP. Investment is much more volatile than consumption. The relative volatility of consumption and investment is comparable to what we find in real data. We simulated the economy over 43 periods, because there were also 43 years of data for German GDP. In the simulation there are nine peaks, which is close to the ten peaks we found in the data. The length of the cycle varies from four to seven years. The average length is a little less than five periods, while the German cycles lasted a little more than four years on average.

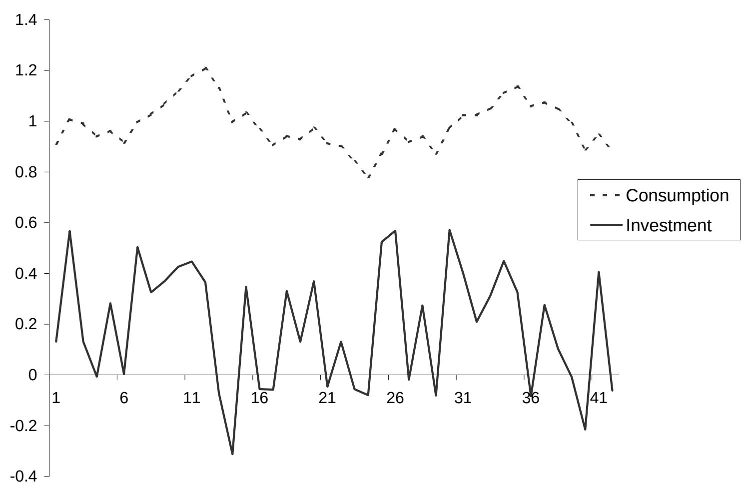


Figure S.5: Simulated Consumption and Investment

Exercise 9.7
The aim of real business cycle research is to gain a better understanding of business cycles. The theory differs from other approaches mainly by the methods that are applied. Real business cycle models are fully specified stochastic equilibrium models. That means that the microfoundations are laid out in detail. There are consumers with preferences, firms

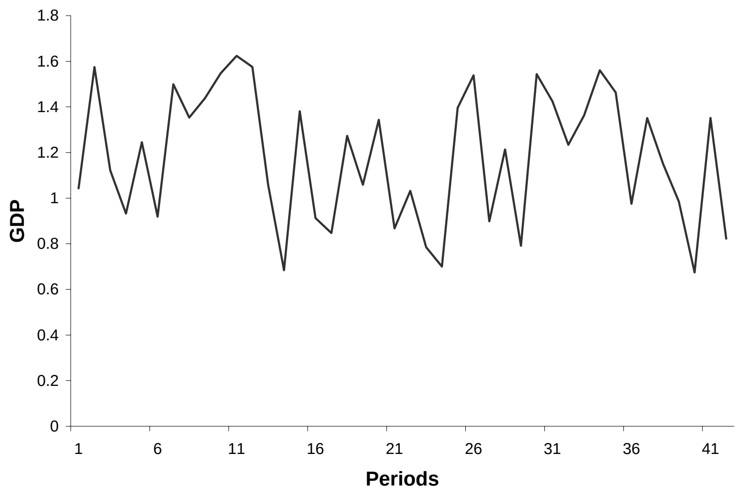


Figure S.6: Simulated GDP

with technologies, and a market system that holds everything together. Real business cycle theory takes the simplest models of this sort as a point of departure to explain business cycles. Model testing is most often done with the "calibration" method. This means that first the model parameters are determined by making them consistent with empirical facts other than the business cycle facts that are supposed to be explained. The parameterized model is then simulated, and the outcomes are compared with real world data.

Exercise 9.8

Plosser's model does not contain a government, and even if there were one, there would be no need to stabilize the economy. There are no market frictions in the model; the outcomes are competitive equilibria. By the First Welfare Theorem we know that equilibria are efficient, so there is nothing a government could do to improve economic outcomes. It is possible to extend the model to allow for a government, and we could add frictions to the model to make intervention beneficial, without changing the general framework very much. Also, any government is certainly able to produce additional shocks in the economy. Still, real business cycle theory works fine without a government, both as a source of disturbance and as a possible stabilizer.

Exercise 10.1

1. This is the myth of small business job creation again. The SBA has every reason to tout the influence of small small businesses, but, as DHS point out, the dominant job market role if played by large, old firms and plants.

2. This rather entertaining quote has several immediate and glaring errors, but it does contain an argument quite in vogue at the moment. There is a common idea that

jobs are a scarce resource, and that the pool of jobs is shrinking under pressure from greedy company owners, slave labor factories abroad and so on. In reality, as we've seen in this chapter, the pool of jobs is churning all the time. Ten percent of all jobs are typically destroyed in a year, and ten percent are created. In the face of this turmoil, one or two high profile plant closings is simply not important.

Exercise 10.2
The term g_{est} is defined as:

$$g_{est} = \frac{X_{es,t} - X_{es,t-1}}{0.5(X_{es,t} + X_{es,t-1})}.$$

For a new plant $X_{es,t-1} = 0$ and for a dying plant $X_{es,t} = 0$. Thus for a new plant:

$$g_{est} = \frac{X_{es,t} - 0}{0.5(X_{es,t} + 0)} = 2.$$

And for a dying plant:

$$g_{est} = \frac{0 - X_{es,t-1}}{0.5(0 + X_{es,t-1})} = -2.$$

Exercise 10.3
The only thing tricky about this problem is remembering how to deal with absolute values. If $a = b$, then $|a| = b$ if a is positive and $|a| = -b$ if a is negative. For shrinking plants, $\Delta X_{es,t}$ is negative, so for shrinking plants:

$$g_{est} = \frac{\Delta X_{es,t}}{Z_{est}} = -\frac{|\Delta X_{es,t}|}{Z_{est}}.$$

Now we work through the algebra required to get the answer to the first identity. We begin with the definition of c_{st}:

$$c_{st} = \frac{C_{st}}{Z_{st}} = \frac{1}{Z_{st}} \sum_{e \in S^+} \Delta X_{es,t} = \frac{1}{Z_{st}} \sum_{e \in S^+} \frac{Z_{est} \Delta X_{es,t}}{Z_{est}} = \frac{1}{Z_{st}} \sum_{e \in S^+} Z_{est} g_{est}.$$

Turning to the next identity, we begin with the definition of net_{st}:

$$\begin{aligned}
\text{net}_{st} &= \frac{C_{st} - D_{st}}{Z_{st}}, \\
&= \frac{1}{Z_{st}} \sum_{e \in S^+} \Delta X_{es,t} - \frac{1}{Z_{st}} \sum_{e \in S^-} |\Delta X_{es,t}|, \\
&= \frac{1}{Z_{st}} \sum_{e \in S^+} Z_{est} g_{est} - \frac{1}{Z_{st}} \sum_{e \in S^+} Z_{est}(-g_{est}), \\
&= \frac{1}{Z_{st}} \sum_{e \in S} Z_{est} g_{est}.
\end{aligned}$$

Exercise 10.4

This question really just boils down to plugging the definitions of R_t and NET_{st} into the definition of covariance. However, the algebra shouldn't detract from an interesting statistical regularity. Begin with the definition of covariance (supplied in the question):

$$\text{cov}(R_t, NET_t) < 0.$$

$$\frac{1}{N}\sum_{i=1}^{N}(R_i - \overline{R})(NET_i - \overline{NET}) < 0.$$

$$\frac{1}{N}\sum_{i=1}^{N}(C_i + D_i - \overline{C} - \overline{D})(C_i - D_i - \overline{C} + \overline{D}) < 0.$$

$$\frac{1}{N}\sum_{i=1}^{N}\left[(C_i - \overline{C}) + (D_i - \overline{D})\right]\left[(C_i - \overline{C}) - (D_i - \overline{D})\right] < 0.$$

$$\frac{1}{N}\sum_{i=1}^{N}(C_i - \overline{C})^2 - \frac{1}{N}\sum_{i=1}^{N}(D_i - \overline{D})^2 < 0.$$

Using the definition definition of variance supplied in the question, this last inequality can be written $\text{var}(C) - \text{var}(D) < 0$, so $\text{var}(C) < \text{var}(D)$. That was a lot of algebra, but it was all straightforward. Thus if periods of large net job loss coincide with periods of larger than normal job reallocation, it must be the case that job destruction has a higher variance than job creation.

Exercise 10.5

Here is the original chart, now augmented with the answers.

Year	$X_{1,t}$	$X_{2,t}$	$X_{3,t}$	c_t	d_t	net_{st}	UB	LB
1990	1000	0	500					
1991	800	100	800	0.250	0.125	0.125	600	200
1992	1200	200	700	0.263	0.053	0.210	600	400
1993	1000	400	600	0.098	0.146	-0.048	500	100
1994	800	800	500	0.195	0.146	0.049	700	100
1995	400	1200	600	0.233	0.186	0.047	900	100
1996	200	1400	600	0.091	0.091	0	400	0
1997	0	2000	500	0.255	0.128	0.127	900	300

Exercise 10.6
All of these statements referred to specific charts or graphs in DHS. This question was on the Spring 1997 midterm exam in Econ 203.

1. Most students were at least able to say that this hypothesis wasn't exactly true, even if they couldn't identify specifically why. Any two of the following facts were acceptable:
   (a) Even the industries in the highest important ratio quintile had an import penetration rate of about 13.1%, which is pretty low.
   (b) The relationship between import penetration quintile and net job growth and job destruction is not monotone.
   (c) For the highest import penetration quintile, net job growth averaged −2.8% annually.
2. Robots replacing workers is another favorite canard (thankfully less common recently) of the chattering classes. The reality is reflected in DHS Table 3.6 showing gross job flows by capital intensity decile. The most fascinating part of this table is the final entry, showing an average annual *net* employment growth rate of 0.7% for plants in the highest capital intensity decile. Plants in the lowest capital intensity decile shed about 10% of their jobs, net, each year. That is, over the 15-year sample period, they must have become nearly extinct. Thus high capital plants (plants with lots and lots of robots, one presumes) have been steadily adding excellent jobs of the past 20 years.
3. What we were looking for here was some version of Figure 2.2 in DHS, giving the distributions of plant-level job creation and destruction by employment growth g. They have a distinctive "double hump" shape with the first peak at about $g = 0.10$. However, we accepted more general statements about how most destruction occurs at plants which are shutting down and so on.
4. This question is drawn directly from Table 3.6, showing that highly specialized plants have high job creation and destruction rates, and a net growth rate of -2%. Because of their high job destruction rate, and the tendency of plants to close in recessions (the cyclical behavior of job destruction), highly specialized plants are indeed at risk of closing in recessions.
5. For this question we wanted students to tell us about job creation and destruction rates by wage quintile (Table 3.4 in DHS). Any two of the following facts were acceptable:
   (a) Job creation and destruction are falling by plant wage quintile.
   (b) Of all jobs destroyed each year, only about 18% are accounted for by the highest wage quintile, while about 26% are accounted for by the lowest wage quintile.
   (c) High wage jobs tend to be more durable (longer creation persistence).

Exercise 11.1

The aggregate production technology is $Y = 3L^{.7}K^{.3}$, and we have $L = 150$, $\delta = 0.1$, and $s = 0.2$. The law of motion for capital is given by:

$$K_t = (1 - \delta)K_{t-1} + sY_t.$$

Therefore the steady state level of capital $\bar{K}$ has to satisfy:

$$\bar{K} = (1 - \delta)\bar{K} + s(3L^{0.7}\ \bar{K}^{0.3}).$$

Plugging in the values for labor, depreciation, and the saving rate yields:

$$\bar{K} = 0.9\bar{K} + (0.6)(150)^{0.7}\ \bar{K}^{0.3},\ \text{or:}$$

$$0.1\bar{K} = (0.6)(150)^{0.7}\ \bar{K}^{0.3},\ \text{or:}$$

$$\bar{K} = \left[(10)(0.6)(150)^{0.7}\right]^{1/0.7}.$$

Evaluating this expression results in $\bar{K} \approx 1940$. Steady state output $\bar{Y}$ is given by:

$$\bar{Y} = 3L^{0.7}\ \bar{K}^{0.3},$$

which gives us the solution $\bar{Y} \approx 970$.

Exercise 11.2

In terms of the Solow model, the war temporarily reduced the capital stock in Kuwait. Given the lower capital stock, per capita incomes will be lower in the next years. In the long run, the economy reaches the steady state again, so the war does not affect per capita income any more. Similarly, the effect on the growth rate of per capita income is also temporary. In the short run, the growth rate will be higher, because the growth rate of per capita income is inversely related to the capital stock. In the long run, the growth rate of per capita income is determined by the rate of technological progress, so the war does not have an effect on the long-run growth rate. Recovery will be faster if foreigners are allowed to invest, because more investment implies that the economy returns faster to the steady state level of capital. The gains and losses of workers and capitalists depend on the reaction of wages and the return on capital to a higher capital stock due to foreign investment. Our formulas for wage and interest, equations (11.3) and (11.4), indicate that the wage is positively related to the capital stock, while the return to capital is negatively related to the capital stock. Since a prohibition of foreign investment lowers the capital stock, workers would lose, and capitalists would gain by a prohibition.

Exercise 12.1

1. True. Under an unfunded pension system payments to the old are made by taxing the young, *not* by investing in the bond market. Hence the volume of physical savings between periods of life is higher under a funded than an unfunded pension system.

2. Check the *Economic Report of the President* to get a good sense of n, and the back of the *Economist* magazine to get the latest value for r. Unless something very odd is happening, n is probably considerably lower than r.

3. From the *Economic Report of the President* we see that (among others) the U.S government spends more than 20% of its total outlays on interest payments on the Federal debt, social security and defense. We shall have quite a bit more to say about the Federal debt in Chapter 14 and Chapter 18.

Exercise 12.2

The household's budget constraint is:

$$C + I + G = Y^P + Y^G.$$

We are given that private output Y^P is fixed at Y and that government output Y^G is ϕG. Thus government spending G must satisfy:

$$\begin{aligned} G &= Y + \phi G - C - I, \text{ or:} \\ (1-\phi)G &= Y - C - I. \end{aligned}$$

Obviously, as G grows, C and I are going to have to shrink (although not one-for-one with G). The maximum allowed level for government spending occurs when consumption and investment are each zero, so $C = I = 0$. In that case:

$$G = \frac{Y}{1-\phi}.$$

The government can spend more than total private output since its spending is productive. As ϕ is closer to zero, the closer G must be to Y. As ϕ is closer to unity, the larger G may be relative to Y.

Exercise 12.3

To calculate the market-clearing interest rate, we have to find the interest rate that makes the household want to consume precisely its endowment stream *net of government taxes*. Since in this question consumption in each period t must just be $C_t = Y_t - G_t$, we find that:

$$\begin{aligned} 1 + r_0 &= \frac{1}{\beta}\frac{U'(Y-G)}{U'(Y-G)}, \text{ so:} \\ r_0 &= \rho, \text{ and:} \\ 1 + r_0^* &= \frac{1}{\beta}\frac{U'(Y-G_0)}{U'(Y-G_1)}. \end{aligned}$$

We cannot characterize r_0^* further without more information about Y, G and U, but we can say that, since $G_0 > G_1$, the marginal utility in the first period must be greater than the marginal utility in the second period, that is, $U'(Y-G_0) > U'(Y-G_1)$. Thus:

$$\frac{U'(Y-G_0)}{U'(Y-G_1)} > 1.$$

As a result, $r_0^* > r_0$. This fits well with the results of this chapter, which hold that temporary increases in government spending increase the real interest rate.

Exercise 12.4
The household's maximization problem becomes:

$$\max_{S_t} \left\{ 2\sqrt{(1-\tau)y - S_t} + 2\beta\sqrt{(1+r)S_t + (1+n)\tau y} \right\}.$$

The first-order condition with respect to S_t is:

$$-\frac{1}{\sqrt{(1-\tau)y - S_t}} + \frac{\beta\sqrt{1+r}}{\sqrt{(1+r)S_t + (1+n)\tau y}} = 0.$$

Solving for S_t produces:

$$S_t = \frac{\beta^2(1+r)}{1+\beta^2(1+r)}y - \frac{\beta^2(1+r) + \frac{1+n}{1+r}}{1+\beta^2(1+r)}\tau y.$$

Notice that private savings is (as usual) decreasing in τ. Also notice that the larger n is relative to r, the greater this effect. When $n > r$, contributions to the social security system supplant private savings at a greater rate than in a funded system. The reason is because, when $n > r$, the social security system is more attractive than private savings.

Exercise 12.5

1. Grace trades consumption today for consumption tomorrow via schooling S (since there is no bond market). Her maximization problem is:

   $$\max_{S} \{\ln(1-S) + \beta \ln(AS)\}.$$

   Recall that $\ln(ab) = \ln(a) + \ln(b)$. Hence the first-order condition is:

   $$-\frac{1}{1-S} + \frac{\beta}{S} = 0.$$

   Solving for Grace's optimal schooling provides $S = \beta/(1+\beta)$. In this setup, $K_1 = S$.

2. Now Grace's problem becomes:

   $$\max_{S} \{\ln(1-S-G) + \beta \ln[A(S+\phi G)]\}.$$

   The first-order condition for maximization is:

   $$-\frac{1}{1-S-G} + \frac{\beta}{S+\phi G} = 0.$$

   Thus Grace's optimal schooling choice becomes:

   $$S = \frac{\beta}{1+\beta} - \frac{\beta+\phi}{1+\beta}G.$$

Grace's schooling is certainly decreasing in G (thus investment is, to a certain extent, being crowded out). Grace's human capital is $K_1 = S + \phi G$, so substituting in provides:

$$K_1 = \frac{\beta}{1+\beta} + (\phi - 1)G\frac{\beta}{1+\beta}.$$

Notice that if $\phi < 1$, the government is less efficient at providing schooling than the private sector, and Grace's human capital decreases in G.

3. Now Grace's maximization problem becomes:

$$\max_S \left\{\ln(1-S) + \beta\ln[A(S+\phi G)]\right\},$$

since Grace does not have to pay a lump-sum tax in the first period. The first order condition is now:

$$-\frac{1}{1-S} + \frac{\beta}{S+\phi G} = 0.$$

Grace's optimal schooling choice is:

$$S = \frac{\beta}{1+\beta} - \phi G\frac{1}{1+\beta},$$

and her human capital becomes:

$$K_1 = \frac{\beta}{1+\beta}(1+\phi G).$$

Notice that Grace's schooling is still being crowded out, but that her human capital is increasing in G no matter what the value of ϕ, as long as $\phi > 0$.

Exercise 13.1

1. If the agent works, $c^i(\ell^i = 1) = 1 - \tau$, while if the agent does not work, $c^i(\ell^i = 0) = 0$.

2. If the agent works, $u^i(\ell^i = 1) = 1 - \tau - \gamma^i$ while if the agent does not work, $u^i(\ell^i = 0) = 0$. An agent will work if the utility of working is greater than the utility of not working, or if $1 - \tau - \gamma^i \geq 0$.

3. From our previous answer, it is easy to see that $\gamma^*(\tau) = 1 - \tau$.

4. We know that the fraction of agents with γ less than or equal to some number, say γ^*, is just γ^* if $0 \leq \gamma^* \leq 1$. Thus aggregate labor supply as a function of the tax rate is just $\ell(\tau) = \gamma^*(\tau) = 1 - \tau$. On each agent who works, the government collects revenue τ. Thus $\mathcal{T}(\tau) = \tau(1-\tau)$. This is sketched in Figure (13.1).

5. There is a Laffer curve in the tax system.

Exercise 13.2
Briefly, although such a result might be evidence for a Laffer curve, the regression does not control for changes in real income over time. There may not truly be a Laffer curve, but it would look like there was one if real incomes were high when taxes were low and low when taxes were high.

Exercise 13.3
The point of this simple problem was to clear up the difference between the *tax system* $\mathcal{H}(a;\psi)$ and the government's *revenue function* $\mathcal{T}(\psi)$. This problem should also give you some practice in thinking about exemptions.

1. The parameters of the tax system are the choices of the flat tax rate τ and the lump-sum tax S. The household chooses an effort level L in response. Thus $\psi = [\tau, S]$ and $a = L$ here.

2. The tax system $\mathcal{H}(a;\psi)$ maps household actions a and tax system parameters ψ into an amount of tax:

   $$\mathcal{H}(L;[\tau,S]) = S + \tau(L - S).$$

   Recall that income directed towards the lump-sum tax S is exempt from the flat tax. We do not consider (yet) that L is itself a function of S and τ.

3. A household's tax bill is always the same as the tax system. In this case, if the household works an amount L it owes $S + \tau(L - S)$.

4. The household's income as a function of L is just L. Hence the household consumes $L - \mathcal{H}(L;[\tau,S])$ or:

   $$C = L - [S + \tau(L - S)] = (1 - \tau)(L - S).$$

5. Substituting in to the household's utility function gives:

   $$U(C,L) = 2\sqrt{(1-\tau)(L-S)} - L.$$

   The first-order condition for maximization with respect to L is:

   $$\frac{1-\tau}{L-S} = 1.$$

   (Where did the 2 go?) Solving for L produces:

   $$L(\tau, S) = S + 1 - \tau.$$

   This gives the household's optimal response to the tax system $\mathcal{H}$. In the chapter we called this $a_{\max}(\psi)$.

6. The government revenue function is the tax system with the household's action a optimized out. That is:

$$\mathcal{T}(psi) = \mathcal{H}\left[a_{\max}(\psi); \psi\right].$$

In this case, this produces:

$$\mathcal{T}([\tau, S]) = \tau(1-\tau) + S.$$

Notice that there is a Laffer curve (as expected) in the tax parameter τ.

For further practice: Assume that income spent on the lump-sum tax is no longer exempt from the flat tax. How do your answers change? You should be able to show that the Laffer curve in τ vanishes.

Exercise 13.4

1. If the household works ℓ, it raises gross income of ℓ and must pay a tax bill of $\tau\ell$. It consumes the residual, $(1-\tau)\ell$.

2. Substitute $c(\ell, \tau)$ into the household's utility function to find utility purely as a function of labor effort. The household's maximization problem becomes:

$$\max_{\ell} \left\{ 4\sqrt{(1-\tau)\ell} - \ell \right\}.$$

Taking the derivative with respect to ℓ gives the first order condition for maximization:

$$2\sqrt{\frac{1-\tau}{\ell}} - 1 = 0.$$

We can solve this to find the household's optimal choice of labor effort given taxes, $\ell(\tau)$:

$$\ell(\tau) = 4(1-\tau).$$

3. That government's tax revenue is:

$$\mathcal{T}(\tau) = \tau\ell(\tau) = 4\tau(1-\tau).$$

4. The government wishes to raise revenue of $3/4$. We are looking for the tax rate τ that satisfies:

$$\begin{aligned} \mathcal{T}(\tau) &= 3/4, \text{ or:} \\ 4\tau(1-\tau) &= 3/4. \end{aligned}$$

Inspection reveals that there are two such tax rates: $\{1/4, 3/4\}$. Since the government is nice, it will choose the lower tax rate, at which the household consumes more.

Exercise 13.5
In the question, you were allowed to assume that $r = 0$ and that Tammy had an implicit discount factor of $\beta = 1$. These solutions are a little more general. To check your solutions, substitute $\beta = 1$ and $r = 0$.

1. In the first period of life, Tammy earns an income of $y = w\ell$ of which she must pay $w\ell\tau_1$ in taxes. Thus her income net of taxes is $w\ell(1 - \tau_1)$. She splits this between consumption in the first period of life, c_1 and savings, b. Thus:

$$\begin{aligned} c_1 + b &\leq (1 - \tau_1)w\ell, \text{ and:} \\ c_2 &\leq (1 + r)b. \end{aligned}$$

Notice that $b = c_2/(1 + r)$ so we can collapse the two one-period budget constraints into a single present-value budget constraint. Thus:

$$c_1 + \frac{1}{1+r}c_2 \leq (1 - \tau_1)w\ell.$$

2. Tammy's Lagrangian is:

$$\mathcal{L}(c_1, c_2, \ell) = \sqrt{c_1} + \beta\sqrt{c_2} - \ell + \lambda\left((1 - \tau_1)w\ell - c_1 - \frac{1}{1+r}c_2\right).$$

This has first-order conditions with respect to c_1, c_2, and ℓ of:

$$\begin{aligned} \frac{1}{2}\frac{1}{\sqrt{c_1}} - \lambda &= 0, \\ \frac{\beta}{2}\frac{1}{\sqrt{c_2}} - \frac{1}{1+r}\lambda &= 0, \text{ and:} \\ -1 + \lambda w(1 - \tau_1) &= 0. \end{aligned}$$

Manipulating each of these equations produces the system:

$$\begin{aligned} c_1 &= \frac{1}{4}\left(\frac{1}{\lambda}\right)^2, \\ c_2 &= \frac{1}{4}\left(\frac{(1+r)\beta}{\lambda}\right)^2, \text{ and:} \\ \frac{1}{\lambda} &= (1 - \tau_1)w. \end{aligned}$$

We can further manipulate these three equations, by substituting out the multiplier λ to find the optimal choices of consumption:

$$\begin{aligned} c_1 &= \frac{1}{4}(1 - \tau_1)^2 w^2, \text{ and:} \\ c_2 &= \frac{1}{4}((1+r)\beta)^2(1 - \tau_1)^2 w^2. \end{aligned}$$

We can find labor effort ℓ by substituting the optimal consumption decisions (calculated above) into the budget constraint. This will tell us how many hours Tammy must work in order to earn enough (after taxes) to afford to consume c_1, c_2. The budget constraint is:

$$\begin{aligned} w(1-\tau_1)\ell &= c_1 + \frac{1}{1+r}c_2 \\ &= \frac{w^2(1-\tau_1)^2}{4}\left(1 + \frac{(1+r)^2\beta^2}{(1+r)}\right) \\ &= \frac{w^2(1-\tau_1)^2}{4}(1+(1+r)\beta^2), \text{ so:} \\ \ell &= \frac{w(1-\tau_1)}{4}(1+(1+r)\beta^2). \end{aligned}$$

Notice that Tammy's effort is strictly decreasing in τ_1 and that at $\tau_1 = 1$, $\ell = 0$. In other words, if the government taxes Tammy to the limit, we expect her not to work at all. This will induce a Laffer curve.

Once we've figured out how much Tammy works, it's an easy matter to deduce how much revenue the government raises by taxing her. The government revenue function here is:

$$\mathcal{T}_1(\tau_1) = \tau_1 w\ell = \frac{w^2\tau_1(1-\tau_1)}{4}[1+(1+r)\beta^2].$$

In terms of τ_1, this is just the equation for a parabola:

$$\mathcal{H}_1(\tau_1) = \tau_1(1-\tau_1)(\text{constant term}).$$

Hence $\tau_1^* = 1/2$, and:

$$\mathcal{H}_1(\tau_1^*) = \left(\frac{1}{4}\right)(\text{constant term}) = \left(\frac{w^2}{16}\right)\left[1+(1+r)\beta^2\right].$$

Thus there is a strict limit on the amount of revenue that the government can squeeze out of Tammy. As the tax rate τ_1 increases, Tammy works less, although if $\tau_1 < 1/2$, the government collects more revenue.

3. There is indeed a Laffer curve in this problem. We should have expected it the instant we saw how Tammy's hours worked, ℓ, responded to the tax rate.

Exercise 13.6

Now Tammy is allowed to deduct savings held over for retirement. This is also known as being able to save in "pre-tax dollars." Almost all employers feature some kind of tax-sheltered savings plan.

1. Tammy's tax bill at the end of period 1 is $\tau_2(wn - b)$. Tammy faces a sequence of budget constraints:

$$\begin{aligned} c_1 &= (1-\tau_2)(w\ell - b), \text{ and:} \\ c_2 &= (1+r)b. \end{aligned}$$

Once again, we use the trick of $b = c_2/(1+r)$, so that Tammy's present-value budget constraint becomes:

$$c_1 = (1-\tau_2)\left(w\ell - \frac{c_2}{R}\right), \text{ so:}$$
$$c_1 + (1-\tau_2)\frac{1}{1+r}c_2 = (1-\tau_2)w\ell$$

Note that as $\tau_2 \to 1$, Tammy's ability to consume in the first period of life goes to zero, but her ability to consume in the second period of life is unchanged.

2. Tammy's Lagrangian is:

$$\mathcal{L}(c_1, c_2, \ell) = \sqrt{c_1} + \beta\sqrt{c_2} - \ell + \lambda\left((1-\tau_2)w\ell - c_1 - \frac{(1-\tau_2)}{1+r}c_2\right).$$

The first-order conditions with respect to c_1, c_2 and ℓ are:

$$\frac{1}{2}\frac{1}{\sqrt{c_1}} - \lambda = 0,$$
$$\frac{\beta}{2}\frac{1}{\sqrt{c_2}} - \frac{1-\tau_2}{1+r}\lambda = 0, \text{ and:}$$
$$-1 + \lambda(1-\tau_2)w = 0.$$

We can write c_1 and c_2 easily as a function of λ:

$$c_1 = \frac{1}{4\lambda^2}, \text{ and:}$$
$$c_2 = \frac{\beta^2}{4\lambda^2}\left(\frac{1+r}{1-\tau_2}\right)^2.$$

So it's an easy matter to substitute out for λ and calculate optimal consumption c_1, c_2. Thus:

$$c_1 = \frac{1}{4}w^2(1-\tau_2)^2, \text{ and:}$$
$$c_2 = \frac{(\beta(1+r))^2}{4}w^2.$$

Notice that c_2 does not depend on τ_2. We can substitute in the optimal consumptions above into the budget constraint to determine how many hours Tammy has to work to be able to afford her optimal consumption plan:

$$\begin{aligned}(1-\tau_2)w\ell &= c_1 + (1-\tau_2)\frac{1}{1+r}c_2 \\ &= \frac{w^2(1-\tau_2)^2}{4} + (1-\tau_2)\frac{\beta^2(1+r)w^2}{4}, \text{ so:} \\ \ell &= \frac{w}{4}[(1-\tau_2) + (1+r)\beta^2].\end{aligned}$$

Notice from the first line above that:

$$w\ell = \frac{c_1}{1-\tau_2} + b.$$

The trick here is to substitute back into the right budget constraint. The first couple of times I did this I substituted back into the budget constraint from Exercise (13.5) and got all sorts of strange answers. Notice that Tammy always consumes a certain amount of c_2, no matter what τ_2 is, so she always works a certain amount. However this may not overturn the Laffer curve since she is paying for c_2 with pre-tax dollars.

3. Once again, this is a bit tricky. Remember that Tammy's tax bill is $\tau_2(w\ell - b)$ and that $b = c_2/(1+r)$. Thus:

$$\begin{aligned}
\mathcal{T}_2(\tau_2) &= \tau_2(w\ell - b) \\
&= \tau_2\left(w\ell - \frac{c_2}{1+r}\right) \\
&= \tau_2\left(w\ell - \frac{\beta^2(1+r)}{4}w^2\right) \\
&= \tau_2\left[w\left(\frac{w}{4}(1-\tau_2) + \frac{w}{4}\beta^2(1+r)\right) - \beta^2(1+r)\frac{w^2}{4}\right] \\
&= \tau_2(1-\tau_2)\left(\frac{w^2}{4}\right).
\end{aligned}$$

This was a matter of remembering to substitute into the right revenue equation. Although Tammy always works at least enough to finance a certain amount of consumption while old, this amount of income is tax-deductible, so the government can't get at it.

As $\tau_2 \to 1$, government revenue goes to zero, as before. The maximizing tax rate, τ_2^*, is $\tau_2^* = 0.5$ and the maximum amount of revenue that government can raise is:

$$\mathcal{T}_2(\tau_2^*) = \frac{w^2}{16}.$$

Notice that $\mathcal{T}_2(\tau_2) < \mathcal{T}_1(\tau_1)$.

4. There is still a Laffer curve present. Unfortunately for the government, tax revenue is now lower.

5. Our answers are indeed different. Because Tammy is able to shelter some of her income from the government, total tax revenue will be lower.

Exercise 14.1

1. About \$5.2 trillion/\$7 trillion.

2. About 1.07 in 1945 (Barro p.362).

3. About 0.38 in 1981 (Barro p.341).

4. 3%.

Exercise 14.2
The consumer's problem is thus:

$$\max_{x_1, x_2} \{\ln(x_1) + x_2\}, \text{ subject to:}$$
$$(p_1 + t_1)x_1 + (p_2 + t_2)x_2 = M.$$

The two first-order conditions for this problem are:

$$\frac{1}{x_1} = \lambda(p_1 + t_1), \text{ and:}$$
$$1 = \lambda(p_2 + t_2).$$

These first order conditions plus the budget constraint can be used to solve for the three unknowns λ, x_1, and x_2 in terms of the givens in the problem, p_1, p_2, t_1, t_2, and M. Solving:

$$x_1 = \frac{p_2 + t_2}{p_1 + t_1}.$$
$$x_2 = \frac{M}{p_2 + t_2} - 1.$$

The government's revenue function can be calculated accordingly:

$$\mathcal{T}(t_1, t_2, p_1, p_2, M) = t_1 x_1 + t_2 x_2 = \frac{t_1(p_2 + t_2)}{p_1 + t_1} + \frac{t_2 M}{p_2 + t_2} - t_2.$$

Substitute the above demand functions for x_1 and x_2 into the objective function to obtain the household's indirect utility:

$$V(p_1 + t_1, p_2 + t_2, M) = \ln\left(\frac{p_2 + t_2}{p_1 + t_1}\right) - \frac{M}{p_2 + t_2} + 1.$$

So at this point, we have found the government's revenue function which tells us how much the government can raise from taxes *given* that consumers respond optimally to the given tax rates. By deriving the consumer's indirect utility function, we know how consumers compare different tax rates and income levels in utility terms.

Potentially the government is faced with the need to raise a certain level of revenue, G. It can raise this revenue a number of different ways by taxing the two goods in different amounts with the constraint that in the end, it must have raised G in revenues.

A benevolent government could decide to choose the combination of taxes (t_1, t_2) such that consumer utility is maximized, subject to the constraint that it raises the necessary revenue G. The government's optimal-tax problem would then be:

$$\max_{t_1,t_2} V(p_1 + t_1, p_2 + t_2, M), \text{ subject to:}$$
$$\mathcal{T}(t_1, t_2, p_1, p_2, M) = G,$$

and some given M. Make sure you understand the intuition of this problem. Both the indirect utility function and the government revenue function account for the fact that households respond optimally to the given tax policy. Before you ever write down the optimal tax problem, you must know how consumers will respond to any possible tax policy given by (t_1, t_2). Implicit in the indirect utility function and government revenue function is the fact that consumers are responding optimally to their environment.

Exercise 14.3

The key to this problem is realizing that the household's budget set will be kinked at the point $\{y_1 - \mathcal{T}_1, y_2 - \mathcal{T}_2\}$. For points to the left of this kink, the household is saving, and the budget set is relatively flat. For points to the right of this kink, the household is borrowing, and the budget set is relatively steep. The government's optimal plan will be to levy very low taxes initially and then high taxes later, in essence borrowing on behalf of the household.

1. If the household neither borrows nor lends, it consumes:

$$c_1 = y_1 - \mathcal{T}_1, \text{ and:}$$
$$c_2 = y_2 - \mathcal{T}_2.$$

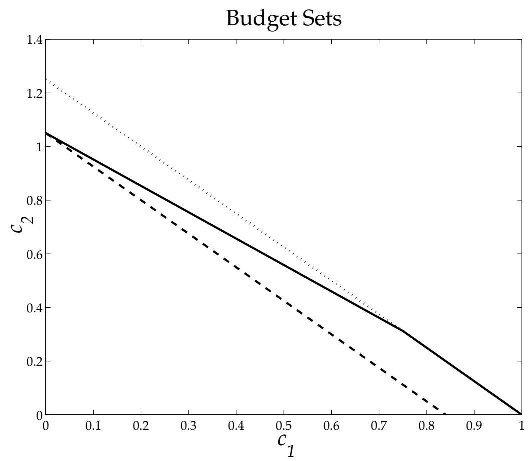
   This is the location of the kink in the budget constraint: to consume more in period $t = 1$ than $y_1 - \mathcal{T}_1$, it will have to borrow at the relatively high rate r' and the budget set will have a slop of $-(1+r')$. The government will be able to move the kink around, increasing or decreasing the number of points in the household's budget set.

2. For convenience, all of the answers to the next three questions are placed on the same set of axes (below). The solid line gives the answer to the first question.

3. The dotted line gives the answer to this question. Notice that the household has more points to choose from.

4. The dashed line gives the answer to this question. Notice that the household has fewer points to choose from.

5. The government chooses $\mathcal{T}_1 = 0$, in essence borrowing at the low interest rate r on behalf of the household.



Exercise 14.4
In this question the government runs a deficit of unity in the first period (period $t = 0$), because expenditures exceed revenues by exactly unity. In all subsequent periods, government revenues just match direct expenditures in each period, but are not enough to repay the interest cost of the initial debt. As a result, the government will have to continually roll over its debt each period. Under the proposed plan, the government has not backed the initial borrowing with any future revenues, so it does not ever intend to repay its debt. From the government's flow budget constraint, assuming that $B^g_{-1} = 0$:

$$\begin{aligned} B^g_0 &= G_0 - \mathcal{T}_0 = 1. \\ B^g_1 &= 1 + r. \\ B^g_2 &= (1 + r)^2. \\ \vdots &\ \vdots \\ B^g_t &= (1 + r)^t. \end{aligned}$$

So the government debt level is exploding. Substituting in to the transversality condition, we get:

$$\lim_{t\to\infty} (1 + r)^{-t} B^g_t = \lim_{t\to\infty} (1 + r)^{-t}(1 + r)^t = 1.$$

Since the limit does not equal zero, we see that the government's debt plan does not meet the transversality condition.

Exercise 15.1
The Lagrangean is:

$$\mathcal{L} = (c^P_w)^\gamma (c^P_b)^{1-\gamma} + \lambda[m^P - c^P_w p_w + c^P_b p_b].$$

The first-order conditions are:

$$\text{(FOC } c_w^P\text{)} \qquad \gamma(c_w^{P\star})^{\gamma-1}(c_b^{P\star})^{1-\gamma} + \lambda^\star[-p_w] = 0, \text{ and:}$$
$$\text{(FOC } c_b^P\text{)} \qquad \gamma(c_w^{P\star})^{\gamma}(1-\gamma)(c_b^{P\star})^{-\gamma} + \lambda^\star[-p_b] = 0.$$

Combining these to get rid of $\lambda^\star$ yields:

$$\frac{\gamma(c_w^{P\star})^{\gamma-1}(c_b^{P\star})^{1-\gamma}}{p_w} = \frac{\gamma(c_w^{P\star})^{\gamma}(1-\gamma)(c_b^{P\star})^{-\gamma}}{p_b}, \text{ or:}$$
$$\frac{p_w}{p_b} = \frac{\gamma c_b^{P\star}}{(1-\gamma)c_w^{P\star}}.$$

Solving this for $c_b^{P\star}$ and plugging back into the budget equation gives us:

$$c_w^{P\star}p_w + \left[\frac{(1-\gamma)p_w c_w^{P\star}}{\gamma p_b}\right] p_b = m^P.$$

After some algebra, we get the first result:

$$c_w^{P\star} = \frac{\gamma m^P}{p_w}.$$

When we plug this back into the budget equation and solve for $b^{P\star}$, we get the other result:

$$c_b^{P\star} = \frac{(1-\gamma)m^P}{p_w}.$$

Exercise 15.2

1. See Figure S.7.

2. See Figure S.7.

3. From the graph, we know that $p_w/p_b = 3$. Suppose the relative price is less than 3. Then only Pat will make wine, and supply will be 4 jugs. Plugging 4 jugs into the demand function gives a relative price of $p_w/p_b = 7/2$, but $7/2 > 3$, which is a contradiction, so the equilibrium relative price can't be less than 3.

   By a similar argument, you can show that the equilibrium relative price can't be more than 3.

4. Pat makes (i) 4 jugs of wine and (ii) 0 jugs of beer. Chris makes (iii) 1 jug of wine and (iv) 1 jug of beer.

5. Pat has an absolute advantage in wine production, since $2 < 6$.

6. Pat has a comparative advantage in wine production, since $2/1 < 6/2$. Chris makes wine anyway, since the equilibrium price is so high.

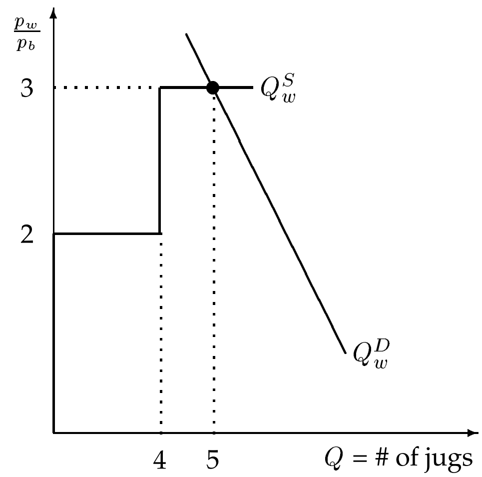


Figure S.7: The Supply of Wine by Pat and Chris Together

Exercise 17.1
We know that:

$$1 + r_2 = F\frac{1 - \theta(1 + r_1)}{1 - \theta}.$$

We can manipulate this to produce:

$$r_2 = F - 1 - \frac{\theta}{1 - \theta}Fr_1.$$

This is an interesting result. Essentially, r_2 is the net return on turnips held until period $t = 2$ (that is the $F - 1$ term) minus a risk premium term that is increasing in r_1.

Exercise 17.2

1. The bank's assets are the value of the loans outstanding net of loss reserve, in other words, the expected return on its loans. The bank's liabilities are the amount it owes its depositors. Considering that the bank must raise a unit amount of deposits to make a single loan, this means that the bank must pay $1 + r$ to make a loan. Thus the bank's expected profits are:

$$\alpha p_S x + (1 - \alpha)p_R x - (1 + r).$$

   That is, the bank gets x only if the borrower does not default.

2. Now we solve for the lowest value of x which generates non-negative expected prof-

its. Setting expected profits to zero produces:

$$\alpha p_S x + (1-\alpha)p_R x = (1+r), \text{ so:}$$
$$x^\star(r,\alpha) = \frac{1+r}{\alpha p_S + (1-\alpha)p_R}.$$

Since safe borrowers repay more frequently than risky ones because:

$$p_S > p_R,$$

the amount repaid, x, is decreasing as the mix of agents becomes safer, that is, as α increases. As expected, x is increasing in the interest rate r.

3. We assumed that agents are risk neutral. Thus if the project succeeds (with probability p_S), a safe agent consumes $\pi_S - x^\star(r,\alpha)$ and a risky agent (with probability p_R) consumes $\pi_R - x^\star(r,\alpha)$. If the project fails, agents consume nothing. Their expected utilities therefore are:

$$V_S(r) = p_S[\pi_S - x^\star(r,\alpha)], \text{ and:}$$
$$V_R(r) = p_R[\pi_R - x^\star(r,\alpha)].$$

Where $x^\star(r,\alpha)$ is the equilibrium value of x. Notice that since $p_S\pi_S = p_R\pi_R$ and $p_S > p_R$ that we can write V_S and V_R as:

$$V_R(r) = V_S(r) + (p_S - p_R)x^\star(r,\alpha).$$

Thus at any given interest rate $r > 0$, the expected utility of risky borrowers is greater than the expected utility of safe borrowers, $V_R(r) > V_S(r)$.

4. Since $V_R(r) > V_S(r)$, it is easy to see that if $V_S(r) > 0$ then $V_R(r)$ must also be greater than zero. Next we find $r^\star$ such that $V_S(r^\star) = 0$. Substituting:

$$0 = V_S(r^\star) = p_S[\pi_s - x^\star(r^\star,\alpha)] = p_S\pi_S - (1+r^\star)\frac{p_S}{\alpha p_S + (1-\alpha)p_R}, \text{ so:}$$
$$1 + r^\star = \pi_S + \alpha\pi_S(p_S - p_R).$$

At interest rates above $r^\star$ all safe agents stop borrowing to finance their projects. Realizing this, banks adjust their equilibrium payments to: $x^\star(r,\alpha = 0)$, so $(1+r)/p_R$.

Exercise 17.3

If the revenue functions $\pi(x,\gamma)$ all shift up by some amount, then, for any given interest rate r, intermediaries can make loans to agents with higher audit costs. That is, $\gamma^\star(r)$ also shifts up as a result. This shifts the demand for capital up and out, but leaves the supply schedule untouched. As a result the equilibrium interest rate increases, as does the equilibrium quantity of capital saved by type-1 (worker) agents. As a result, type-1 agents work harder, accumulate more capital and more type-2 (entrepreneurial) agents' projects are funded so aggregate output goes up. Type-1 agents are made better off by the increase in the interest rate because their consumption goes up (although they are working harder too). Type-2 agents who had been credit rationed are made better off, but type-2 agents who previously had not been credit rationed are made worse off because the interest rate paid on their loans goes up.

Exercise 17.4

This question uses slightly different notation from that used in this chapter. Most bothersome is probably the fact that r here denotes the *gross* interest rate, which elsewhere is denoted $1+r$. This question is a reworking of the model of moral hazard from this chapter. This question was taken directly from the Spring 1998 Econ 203 final exam.

1. A rich Yalie can finance the tuition cost of Yale from her own wealth (that is, $a > 1$). If she gets the good job, she consumes $w + r(a-1)$, if she does not, she consumes $r(a-1)$. Hence her maximization problem is:

$$\max_{\pi} \left\{ \pi[w + r(a-1)] + (1-\pi)[r(a-1)] - \frac{w}{\alpha}\frac{\pi^2}{2} \right\}.$$

The first-order condition with respect to π is:

$$w - \frac{w}{\alpha}\pi = 0.$$

We can easily solve this to find that $\pi = w$.

2. Poor Yalies are required to repay an amount x only if they land the good job. Hence if they land the good job, they consume $w - x$, while if they go unemployed, they consume 0. Thus their optimization problem may be written as:

$$\max_{\pi} \left\{ \pi(w-x) + (1-\pi)\cdot 0 - \frac{w}{\alpha}\frac{\pi^2}{2} \right\}.$$

The first-order condition with respect to effort π is now:

$$w - x - \frac{w}{\alpha}\pi = 0.$$

We can solve this to find the optimal effort as a function of repayment amount:

$$\pi(x) = \alpha\left(1 - \frac{x}{w}\right).$$

Notice that effort is decreasing in x.

3. Yale University must also pay r to raise the funds to loan to its students. If it is making these loans out of its endowment, then it is paying an opportunity cost of r. A student of wealth $a < 1$ needs a loan of size $1-a$, which costs Yale an amount $r(a-1)$. Thus Yale's profit on this loan is:

$$x\pi(x) + 0\cdot[1-\pi(x)] - r(a-1).$$

But we know $\pi(x)$ from the previous question, so:

$$x\alpha\left(1 - \frac{x}{w}\right) - r(a-1).$$

This is the usual quadratic in x.

4. Yale's "fair lending policy" guarantees that *all* borrowers pay the same interest rate, regardless of wealth. Since we know $\pi(x)$ from above, and we are given $x(a)$, it is an easy matter to calculate $\pi(a)$:

$$\pi(a) = \alpha - \frac{r}{w}(1-a).$$

Notice that effort is decreasing in r and increasing in wealth a.

5. Here we are supposed to show that $\pi(a) \leq \pi^\star$ from above, where $\pi^\star = \alpha$. If $a < 1$ then $1 - a > 0$, and $r > 0$ and $w > 0$ by assumption. It's easy to see that this must be true.

6. Now we are supposed to show that Yale's profits are negative on loans and that poor borrowers cost it more than richer borrowers. The fair lending policy charge all borrowers the same interest rate. Further, this interest rate guarantees Yale zero profits assuming that they exert α effort. Poor borrowers will exert less than α effort, and so Yale will lose money. Return to Yale's profit function:

$$\pi(a)x(a) - r(1-a).$$

Substituting in for $\pi(a)$ and $x(a)$ we get:

$$\left[\alpha - \frac{r}{w}(1-a)\right]\left[\frac{r(1-a)}{\alpha}\right] - r(1-a).$$

We can manipulate this to produce:

$$-\frac{[r(1-a)]^2}{\alpha w}.$$

All other terms canceled out. This is certainly negative, and increasing in a. Thus Yale loses no money on "borrowers" of wealth $a = 1$, and loses the most money on borrowers of wealth $a = 0$.

Exercise 18.1

This question has been given on previous problem sets. In particular, we have amassed a few years' data on students currency holding habits.

1. According to Friedman and Schwartz, the stock of money fell 33% from 1929 to 1933. Household holdings of currency increased over the period.

2. Real income fell by 36% over the same period and prices decreased.

3. From the Barro textbook: Real interest rates have been negative in the years 1950-51, 1956-57 and 1973-79. Inflation was negative in 1949 and 1954.

4. From the Barro textbook: There is evidence in looking cross-sectionally at different countries that changes in money stocks are positively correlated with changes in prices, or inflation. Long run time-series evidence demonstrates a positive correlation between money growth and inflation as well.

5. From the Porter article on the location of U.S. currency: The stock of Federal Reserve notes outside of banks (vault cash) at the end of 1995 was about $375 billion, or about $1440 per American. Nobody had quite this much cash on them, although some students were carrying over $100. I assume these students were well trained in self-defense. According to Porter, between $200 and $250 billion, that is, more than half, was abroad, primarily in the former Soviet Union and South America.

6. Generally people keep their money in low interest assets because they are liquid and provide transactions services. It's tough to buy lunch with shares of GM stock rather than Hyde Park bank checks.

7. Sargent states that inflation can seem to have momentum if people have persistent expectations that the government will continue to pursue inflationary fiscal and monetary policies.

8. Since currency is a debt of the government, whenever the government prints money, it is devaluing the value of its debt. This is a form of taxation and the value by which its debt is reduced is called seignorage. The government obtained $23 billion in seignorage in 1991.

9. The quantity theory is the theory that the stock of money is directly related to the nominal value of output in the economy. It is usually written as the identity:

$$M = PY/V$$

where M is the money stock, P is the price level, and Y is the real amount of output. It is an accounting identity in that the velocity of money, V, is defined residually as whatever it takes to make the above identity true.

10. A gold standard is a monetary system where the government promises to exchange dollars for a given amount of gold. If the world quantity of gold changes (for example, gold is discovered in the Illinois high country) then the quantity of money also changes. Our current monetary system is a fiat system, where money isn't backed by any other real asset. It is simply money by "fiat".

Exercise 18.2
Government austerity programs involve reducing government expenditures and increasing tax revenue. Both cause immediate and obvious dislocations. Governments typically reduce spending by firing lots of government workers, closing or privatizing loss-making government-owned industries and reducing subsidies on staples like food and shelter. Governments increase revenue by charging for previously-free services and pushing up the tax rates. From the point of view of a typical household, expenses are likely to go up

while income is likely to fall. Thus austerity programs can indeed cause immediate civil unrest.

On the other hand, we know that subsidies are a bad way to help the poor (since most of the benefit goes to middle-class and rich households), that state-owned businesses tend to be poorly run, depressing the marginal product of workers and tying up valuable capital and that bloated government bureaucracies are rarely beneficial.

Leave all this to one side: the fact is that no government willingly embarks on an austerity program. They only consider austerity when they are forced to choose between austerity and hyperinflation. Like Germany in 1921, an austerity program has to be seen as better than the alternative, hyperinflation. The central European countries in the early 1920s tried both hyperinflation and austerity, and found austerity to be the lesser of the two evils. That early experience has since been confirmed by a host of different countries. Austerity may indeed be painful, but it is necessary in the long run and better than hyperinflation.

Exercise 18.3

1. We know that the money supply must evolve to completely cover the constant per-capita deficit of d. So we know that:

$$\frac{M_t - M_{t-1}}{P_t} = D_t = dN_t. \tag{S.10}$$

We know from the Quantity Theory of Money given in the problem that:

$$P_t = \frac{M_t}{Y_t} = \frac{M_t}{N_t}. \tag{S.11}$$

Thus we can put equations (S.10) and (S.11) together to produce:

$$dN_t = \frac{M_t - M_{t-1}}{\frac{M_t}{N_t}} = N_t \frac{M_t - M_{t-1}}{M_t}, \text{ so:}$$

$$d = \frac{M_t - M_{t-1}}{M_t}, \text{ so:}$$

$$1 - \frac{M_{t-1}}{M_t} = d, \text{ and:}$$

$$\frac{M_t}{M_{t-1}} = \frac{1}{1-d}. \tag{S.12}$$

Thus $M_t = [1/(1-d)]M_{t-1}$. This gives us an expression by how much the total stock of money must evolve to raise enough seignorage revenue to allow the government to run a constant per-capita deficit of d each period.

2. To answer this question we will use the quantity-theoretic relation, equation (S.11) above and the effect of d on the evolution of money in equation (S.12) above to a find a value for d at which prices are stable, that is, at which $P_t = P_{t-1}$. Notice that:

$$\frac{P_t}{P_{t-1}} = \frac{M_t/N_t}{M_{t-1}/N_{t-1}} = \frac{N_{t-1}}{N_t}\frac{M_t}{M_{t-1}} = \frac{1}{1+n}\frac{M_t}{M_{t-1}} = \frac{1}{1+n}\frac{1}{1-d}. \tag{S.13}$$

If $P_t/P_{t-1} = 1$ then, continuing from (S.13):

$$\frac{1}{1+n}\frac{1}{1-d} = 1, \text{ so:}$$

$$d = \frac{n}{1+n}. \tag{S.14}$$

By (S.14) we see that the government can run a constant per capita deficit of $n/(1+n)$ by printing money and not cause any inflation, where n is the growth rate of the economy/population (they are the same thing in this example).

3. As $n \to 0$, the non-inflationary deficit also goes to zero. At $n = 1$ (the economy doubles in size every period) the non-inflationary per-capita deficit goes to $1/2$. That is, the government can run a deficit of 50% of GDP by printing money and not cause inflation. At the supplied estimate of $n = 0.03$, the critical value of d is $0.03/1.03$ or about 0.029 or 2.9% of GDP.

4. From equation (S.13) above, if $d = 0$ then:

$$\frac{P_t}{P_{t-1}} = \frac{1}{1+n} < 1, \text{ so:}$$
$$P_t = \frac{1}{1+n}P_{t-1}, \text{ and:}$$
$$P_t < P_{t-1}.$$

So there will be deflation over time—prices will fall at the rate n.

Exercise 18.4
Although we will accept a variety of answers, I will outline briefly what we were looking for. As with the central European countries in 1921-23, Kolyastan is politically unstable and in economic turmoil. Many of the same policies that worked in those countries should also work in Kolyastan. The government should move quickly to improve its tax collection system and radically decrease spending. This will probably mean closing down state-run factories and ending subsidies. The argument, often advanced, that such direct measures will hurt the citizens ignores the fact that the people are already paying for them through the inefficient means of the inflation tax. With its fiscal house in order, the government should reform the monetary sector by liberating the central bank, appointing a dour old man to be its head and undertaking a currency reform. For these changes to be credible, Kolyastan must somehow commit not to return to its bad old ways. It could do so by signing treaty agreements with the IMF, World Bank or some other dispassionate outside entity. Further, it should write the law creating the central bank in such a way that it is more or less independent from transitory political pressures. The bank ought to be prohibited from buying Kolyastani Treasury notes.

Exercise 19.1
1. True: The CPI calculates the change in the price of a market-basket of goods over fairly short time periods. If one element of that basket were to increase in price dra-

matically, even if they were compensated enough to buy the new market basket, consumers would choose one with less of the newly-expensive good (substituting away from it).

2. Inflation is bad because it leads consumers to undertake a privately useful but socially wasteful activity (economizing on cash balances). The Fed cannot effectively fight inflation with short-term actions, it must maintain a long-term low-inflation regime.

Exercise 19.2
The slope of the Phillips curve gives the relative price (technological tradeoff) between inflation and unemployment. If inflationary expectations are fixed, the government can achieve a higher utility if it does not have to accept more inflation for lower unemployment. In other words, if the Phillips curve is flatter. It is interesting to note that a perfectly flat Phillips curve would mean that unemployment was purely a choice of the government and did not affect inflation at all. If the government and the private sector engage in a Nash game, the Nash outcomes inflation rate is directly proportional to γ, so low values of γ mean lower Nash inflation.

Exercise 19.3
The point of this question was bested summed up by Goethe in Faust. His Mephistopheles at one point describes himself: "That Power I serve / Which wills forever Evil / And does forever good." Or as Nick Lowe put: "You've got to be cruel to be kind." The higher ϕ is the higher the inflation rate, but unemployment is only marginally lower (depending on expectations).

1. The government's maximization problem is:

$$\max_{\pi} \left\{ -\phi(u^* + \gamma\pi^e - \gamma\pi)^2 - \pi^2 \right\}.$$

   We can solve this to find:

$$\pi_0(\phi) = \frac{\phi\gamma}{1+\phi\gamma^2}u^* + \frac{\phi\gamma^2}{1+\phi\gamma^2}\pi^e.$$

   Thus the optimal inflation choice is increasing in ϕ.

2. The corresponding unemployment rate is:

$$u_0(\phi) = \frac{1}{1+\phi\gamma^2}u^* + \frac{\gamma}{1+\phi\gamma^2}\pi^e.$$

3. Now we assume that government continues to take expectations as fixed, but that the private sector adjusts its expectations so that they are perfectly met. Recall that, given expectations π^e, the government's optimal inflationary response is:

$$\pi = \frac{\phi\gamma}{1+\phi\gamma^2}u^* + \frac{\phi\gamma^2}{1+\phi\gamma^2}\pi^e.$$

Now define π_1 as:

$$\pi_1 = \frac{\phi\gamma}{1+\phi\gamma^2}u^* + \frac{\phi\gamma^2}{1+\phi\gamma^2}\pi_1.$$

We can solve this for π_1 to find:

$$\pi_1 = \phi\gamma u^*.$$

The associated unemployment rate is $u_1 = u^*$, since $\pi^e = \pi$ in this case.

4. Given that agents form expectations rationally, eventually π^e will converge to π. If the government is playing Ramsey (because it has a commitment device), then $\pi = 0$ and $u = u^*$ no matter what ϕ is. If the government is playing Nash, then unemployment is still at the natural rate, $u = u^*$, but inflation is $\pi = \phi\gamma u^*$. Thus the lower the value of ϕ, the lower the Nash inflation rate. The point of this question is that if $\phi = 0$, the Nash and Ramsey inflation rates coincide. Having $\phi = 0$ is an effective device with which to commit to low inflation.

Exercise 19.4
Think of the dynamics in this question as sliding along the government's best response curve, as depicted in Figure 19.2. Expectations will creep up, always lagging behind actual inflation, until the gap between the two vanishes and the private sector expects the Nash inflation, and the government (of course) delivers it.

1. In period t, given inflationary expectations π_t^e, the government solves:

$$\max_{\pi_t} \left\{ -\{u^\star + \gamma(\pi_t^e - \pi_t)\}^2 - \pi_t^2 \right\}.$$

The government's optimal choice is:

$$\pi_t^\star(\pi_t^e) = \frac{\gamma}{1+\gamma^2}u^\star + \frac{\gamma^2}{1+\gamma^2}\pi^e.$$

2. Since expectations are just last period's inflation rate, and since we know that the government inflation policy rule is given by $\pi_t^\star$ above, the dynamics of the system are given by the pair of equations:

$$\begin{aligned} \pi_t &= A(u^\star + \gamma\pi_t^e), \text{ for all } t = 0, 1, \ldots, \infty, \text{ and:} \\ \pi_t^e &= \pi_{t-1}, \text{ for all } t = 1, 2, \ldots, \infty. \end{aligned}$$

Recall that initial inflationary expectations are $\pi_0^e = 0$. We can substitute out the expectations term to produce a single law of motion in inflation:

$$\pi_t = Au^\star + \gamma A\pi_{t-1}, \text{ for all } t = 1, 2, \ldots, \infty.$$

For notational convenience we have defined $A = \gamma/(1+\gamma^2)$.

3. Since expectations start at zero, the first period's inflation rate is:

$$\pi_0 = Au^\star.$$

Where is as defined above. Thus in the first few periods inflation evolves as:

$\pi_0 = Au^\star.$
$\pi_1 = Au^\star + \gamma A\pi_0 = Au^\star + \gamma A^2u^\star = Au^\star(1 + \gamma A).$
$\pi_2 = Au^\star + \gamma A\pi_1 = Au^\star + (1 + \gamma A)\gamma A^2u^\star = Au^\star(1 + \gamma A + (\gamma A)^2).$
$\pi_3 = Au^\star + \gamma A\pi_2 = Au^\star + \gamma Au^*(1 + \gamma A + (\gamma A)^2) = Au^\star(1 + \gamma A + (\gamma A)^2 + (\gamma A)^3).$

The pattern ought to be pretty clear. In general, inflation in period t will be:

$$\pi_t = Au^\star \sum_{i=0}^{t} (\gamma A)^i.$$

So as time moves forward, we have:

$$\lim_{t\to\infty} \pi_t = Au^\star \sum_{i=0}^{\infty} (\gamma A)^i.$$

We can solve the summation using the geometric series to get:

$$\lim_{t\to\infty} \pi_t = \frac{A}{1 - \gamma A}u^\star.$$

Recall that we defined A to be:

$$A = \frac{\gamma}{1 + \gamma^2}.$$

So we can further simplify to get:

$$\lim_{t\to\infty} \pi_t = \gamma u^\star.$$

This is just the Nash inflation rate. Expectations are also converging to this level, so at the limit, unemployment will also be at the Nash level of the natural rate $u^\star$.

Given that inflationary expectations were initially low, the government was able to surprise the private sector and push unemployment below its natural level. Over time the private adapted its expectations and as expected inflation rose, so did unemployment. Thus the time paths of inflation and unemployment are both rising over time, until they achieve the Nash level.

4. The steady-state levels of inflation and unemployment are not sensitive to the initial expected inflation. If the private sector were instead anticipating very high inflation levels at the beginning of the trajectory, the government would consistently produce surprisingly low inflation levels (but still above the Nash level) and the unemployment rate would be above its natural rate. Over time both inflation and unemployment would fall to their Nash levels.

5. The government's optimal choice of inflation in period t, π_t, now becomes:

$$\pi_t = A(u^\star + \gamma\pi_{t-1} + \gamma\varepsilon_t), \text{ for all } t = 1, 2, \ldots, \infty.$$

Since the shock term is mean zero, over time we would expected the inflation rate to settle down in expectation to the same level as before, although each period the shock will push the inflation rate above or below the Nash level. In Figure (c19:fa3) we plot the mean and actual trajectories for inflation and unemployment.

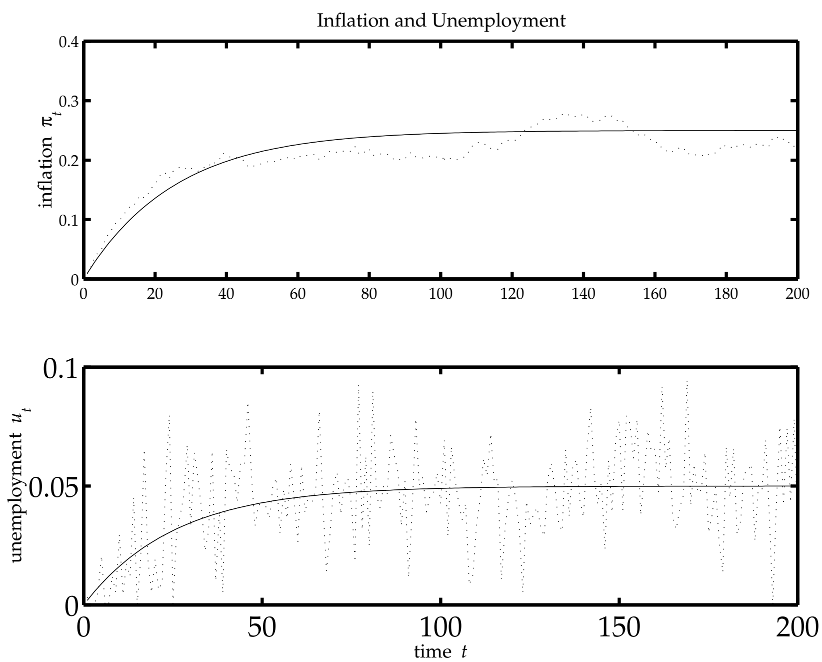


Figure S.8: The dotted line gives the actual time paths for inflation and unemployment with adaptive expectations when there is a mean-zero i.i.d. Normal shock to the Phillips curve, while the solid lines give the same thing with the shock turned off.

Exercise 19.5

As in the previous question, the dynamics of expectations and inflation are given by the system:

$$\pi_t = A(u^\star + \gamma\pi_t^e), \text{ for all } t = 0, 1, \ldots, \infty, \text{ and:}$$
$$\pi_t^e = \delta\pi_{t-1}, \text{ for all } t = 1, 2, \ldots, \infty.$$

Recall that initial inflationary expectations are defined to be $\pi_0^e = 0$. Again, the term A is defined to be:

$$A = \frac{\gamma}{1 + \gamma^2}.$$

We can substitute out the expectations term above to determine the law of motion for inflation:

$$\pi_t = Au^\star + \delta A\gamma\pi_{t-1}, \text{ for all } t = 1, 2, \dots, \infty.$$

Eventually this will converge to a steady-state level of inflation, at which $\pi_{t+1} = \pi_t = \pi_1$. Substituting in:

$$\pi_1 = Au^\star + \delta\gamma A\pi_1.$$

Solving for π_1 produces:

$$\pi_1 = \frac{A}{1 - \delta\gamma A}u^\star.$$

The associated inflation rate, u_1, is:

$$u_1 = \frac{1 - \gamma A}{1 - \delta\gamma A}u^\star.$$

Notice that if $\delta = 1$ this is just the normal Nash outcome. As δ moves closer to zero, so that the private sector puts more and more weight on the government's (utterly mendacious) announcement, inflation and unemployment both fall.