AMBIGUITY AND AMBIGUITY AVERSION

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1. Introduction

Consider the following choice problem, known as "Ellsberg's three-color urn example", or simply the "Ellsberg Paradox" (Ellsberg [7]). An urn contains 30 red balls, and 60 green and blue balls, in unspecified proportions; subjects are asked to compare (i) a bet on a red draw vs. a bet on a green draw, and (ii) a bet on a red or blue draw vs. a bet on a green or blue draw. If the subject wins a bet, she receives 10 dollars; otherwise, she receives 0. To model this situation as a problem of choice under uncertainty, let the state space be $\{s_r, s_g, s_b\}$, in obvious notation, and consider the bets in Table 1.

Table 1. Ellsberg's three-color urn.

	s_r	s_g	s_b
f_r	10	0	0
f_g	0	10	0
f_{rb}	10	0	10
f_{gb}	0	10	10

The modal preferences in this example are $f_r \succ f_g$ and $f_{rb} \prec f_{gb}$, where "$\succ$" denotes strict preference.[1] A common rationalization runs as follows: betting on red is "safer" than betting on green, because the urn may actually contain zero green balls; on the other hand, betting on green or blue is "safer" than betting on red or blue, because the urn may contain zero blue balls. Equivalently, when evaluating f_r and f_{gb}, the fact that the relative likelihood of green vs. blue balls is unspecified is irrelevant; on the other hand, this consideration looms large when evaluating the acts f_g and f_{rb}.

While these preferences seem plausible, they are inconsistent with subjective expected utility maximization (SEU henceforth). Indeed, they are inconsistent with the weaker assumption that the decision-maker's (DM) qualitative beliefs, as revealed by her betting behavior, can be numerically represented by a probability measure. Note that $f_r \succ f_g$ indicates that r is deemed strictly more likely than g, so any probability P that represents the individual's likelihood ordering of events must satisfy $P(\{r\}) > P(\{g\})$; on the other hand, $f_{rb} \prec f_{gb}$ indicates that $\{r, b\}$ is strictly less likely than $\{g, b\}$, which would require $P(\{r\}) + P(\{b\}) = P(\{r, b\}) < P(\{g, b\}) = P(\{g\}) + P(\{b\})$, hence $P(\{r\}) < P(\{g\})$.

The key to Ellsberg's example is the fact that the composition of the urn is incompletely specified: the relative likelihood of a green vs. a blue draw is "ambiguous". More generally, in the words of Daniel Ellsberg, *ambiguity* is

> a quality depending on the amount, type, reliability and 'unanimity' of information, and giving rise to one's 'degree of confidence' in an estimate of relative likelihoods ([7, p. 657]).

Borrowing Ellsberg's terminology, the modal preferences $f_r \succ f_g$ and $f_{rb} \prec f_{gb}$ indicate that the DM would rather have the ultimate outcome of her choices (i.e. whether she receives 10 or 0) depend upon events whose relative likelihood she is more confident about. In other words, these preferences denote *ambiguity aversion.*

Over the last twenty years, several decision models that can accommodate ambiguity and ambiguity aversion (or appeal) have been axiomatized; other contributions have addressed the behavioral manifestations and implications of ambiguity, as well as updating and dynamic choice. Furthermore, there is an ever-growing collection of applications to contract theory, auctions, finance, macroeconomics, political economy, insurance and other areas of economic inquiry.

[1] Ellsberg did not conduct actual experiments, but similar patterns of behavior have been reported in subsequent experimental studies; see Camerer and Weber [4] for an exhaustive survey.

The following Section reviews two of the most influential models of ambiguity-sensitive preferences in a static setting. Section 3 briefly discusses additional models, updating, and dynamic choice.

2. "Classical" Models of Ambiguity-Sensitive Preferences

2.1. **Preliminaries.** Fix a finite or infinite state space S and an algebra Σ of its subsets. A *probability charge* is set function $P : \Sigma \to [0,1]$ that satisfies $P(S) = 1$ and $P(E \cup F) = P(E) + P(F)$ for all $E, F \in \Sigma$ with $E \cap F = \emptyset$; that is, P is normalized and *finitely* additive. The set of probability charges on (S, Σ) is denoted $\Delta(S, \Sigma)$.

The decision models discussed in this section were first axiomatized in the framework introduced by Anscombe and Aumann [2]; it is convenient to adopt the same setup here.[2] Fix a set of prizes X, and let $\Delta(X)$ be the collection of all lotteries (probability distributions) on X with finite support. An act is a Σ-measurable map $f : S \to \Delta(X)$. The set $\Delta(X)$ is closed under mixtures, i.e. convex combinations; mixtures of acts are then defined pointwise, so that the set $\mathcal{F}$ of all acts is also closed under mixtures.[3]

A preference is a binary relation $\succeq$ on $\mathcal{F}$; its symmetric and asymmetric parts are denoted by $\sim$ and $\succ$ respectively. It is customary to identify every lottery $p \in \Delta(X)$ with the constant act that yields p in every state.

A (von Neumann-Morgenstern, or Bernoulli) utility function is a map $u : \Delta(X) \to \mathbb{R}$ that satisfies $u(\alpha p + (1-\alpha)q) = \alpha u(p) + (1-\alpha)u(q)$ for all $\alpha \in [0,1]$ and $p, q \in \Delta(X)$. All axiomatizations discussed below ensure that preferences over lotteries can be represented by a utility function.

A function $a : S \to \mathbb{R}$ is simple if its range is finite; write $a = (a_1, E_1; \ldots; a_n, E_n)$, where $a_1, \ldots a_N \in \mathbb{R}$ and $E_1, \ldots, E_N$ is a partition of S, to indicate that, for all $n = 1 \ldots, N$, $a(s) = a_n$ for all $s \in E_n$. An act is simple if its range can be partitioned into finitely many indifference classes. The set of simple Σ-measurable acts is denoted by $\mathcal{F}_0$.

Virtually all substantive decision-theoretic issues can be analyzed restricting attention to preferences over $\mathcal{F}_0$; the reader is urged to consult the references cited for a discussion of preferences over non-simple acts.

2.2. **Capacities and Choquet-Expected Utility.** The modal preferences in the three-color urn example are inconsistent with a probabilistic representation of beliefs essentially because probabilities are finitely additive. Specifically, if the probability charge P represents the individual's qualitative beliefs, $f_{rb} \prec f_{gb}$ requires that $P(\{r, b\}) < P(\{g, b\})$; since P is additive, this implies $P(\{r\}) < P(\{g\})$. However, $f_r \succ f_g$ implies the reverse inequality. Thus, the Ellsberg paradox can be formally "resolved" if a weaker, non-additive representation of the individual's qualitative beliefs is allowed. This approach is pursued in Schmeidler [24, 25].

A *capacity* is a set function $v : \Sigma \to [0,1]$ such that $v(S) = 1$ and $v(A) \leq v(B)$ for all events $A, B \in \Sigma$ such that $A \subseteq B$. Thus, a capacity is not required to be additive, although it must satisfy a monotonicity property that has a natural interpretation in terms of qualitative beliefs: "larger" events are "more likely".

To define expectation with respect to capacities, a suitable notion of integration is required. Consider a simple function $a = (a_1, E_1; \ldots; a_N, E_N)$, with $a_1 > a_2 > \ldots > a_N$. The *Choquet integral* of a with respect to a capacity v (Choquet [5]) is the quantity

$$\int a \, dP = \sum_{n=1}^{N-1} (a_n - a_{n+1}) v \left(\bigcup_{m=1}^{n} E_m \right) + a_N. \tag{2.1}$$

With the convention that $\cup_{m=1}^{0} E_m = \emptyset$, Eq. (2.1) can be rewritten as follows:

$$\int a \, dP = \sum_{n=1}^{N} a_n \left[v \left(\bigcup_{m=1}^{n} E_m \right) - v \left(\bigcup_{m=1}^{n-1} E_m \right) \right]. \tag{2.2}$$

Thus, Choquet integration performs a "weighted average" of the values $a_1, \ldots, a_N$, with non-negative weights $v(E_1), v(E_1 \cup E_2) - v(E_1), \ldots, 1 - v(E_1 \cup \ldots \cup E_{N-1})$ that add up to one. If v is additive, Eq. (2.1) reduces to $\int a \, dP = \sum_{n=1}^{N} a_n v(E_n)$. However, in general, the ordering of the values $a_1, \ldots, a_N$ affects the decision

[2] Alternative axiomatizations that do not rely on lotteries have also been obtained: see e.g. Gilboa [15], Ghirardato et al. [13] and references therein.

[3] That is: for all $\alpha \in [0,1]$ and all $f, g \in \mathcal{F}_0$, $\alpha f + (1-\alpha) g$ is the act that yields the lottery $\alpha f(s) + (1-\alpha) g(s)$ in state $s \in S$.

weights: for instance, suppose $a = (\alpha, E; \beta, S \setminus E)$, with $\alpha \neq \beta$: then $\int a\, dv$ equals $\alpha v(E) + \beta[1 - v(E)]$ if $\alpha > \beta$, and $\beta v(S \setminus E) + \alpha[1 - v(S \setminus E)]$ if $\beta > \alpha$. These expressions are different unless $v(E) + v(S \setminus E) = 1$.

A preference admits a *Choquet-Expected Utility* (CEU) representation if there exists a utility function u and a capacity v such that, for all simple acts $f, g \in \mathcal{F}_0$, $f \succeq g$ if and only if $\int u(f(s))\, dv \geq \int u(g(s))\, dv$, where the integrals are as in Eq. (2.1).

Preferences in the Ellsberg paradox are consistent with CEU. Let u satisfy $u(10) > u(0)$, and observe that $f_r \succ f_g$ requires $v(\{r\}) > v(\{g\})$, whereas $f_{rb} \prec f_{gb}$ implies that $v(\{r, b\}) < v(\{g, b\})$; since v is not required to be additive, these inequalities can be mutually consistent: for instance, let

$$v(\{r\}) = v(\{r, b\}) = v(\{r, g\}) = \frac{1}{3}, \quad v(\{b\}) = v(\{g\}) = 0, \quad \text{and} \quad v(\{b, g\}) = \frac{2}{3}. \tag{2.3}$$

Recall that the key axiom in the Anscombe-Aumann axiomatization of SEU is *Independence*: for all triples of (simple) acts f, g, h, and all $\alpha \in (0, 1)$, $f \succ g$ implies $\alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$. Schmeidler [25] shows that CEU preferences are instead characterized by a weaker independence property. Say that two acts f and g are *comonotonic* if there is no pair of states s, s' such that $f(s) \succ f(s')$ and $g(s) \prec g(s')$; the key axiom in Schmeidler's characterization of CEU preferences, *Comonotonic Independence*, requires that $f \succ g \Rightarrow \alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$ only if f, g, h are pairwise comonotonic.

To illustrate the rationale behind this weakening of Independence, consider the acts f_r and f_g in the Ellsberg paradox, and define a third act f_b by $f_b(r) = f_b(g) = 0$ and $f_b(b) = 10$. For the CEU preferences defined above, $f_r \succ f_g$, but $\frac{1}{2} f_r + \frac{1}{2} f_b \prec \frac{1}{2} f_g + \frac{1}{2} f_b$. This is consistent with the notion that the DM dislikes ambiguity, and hence would rather have the ultimate outcome of her choices depend upon events whose relative likelihood she is more confident about; in particular, notice that the mixture $\frac{1}{2} f_g + \frac{1}{2} f_b$ yields the same outcome in states g and b, so the DM need not worry about her lack of confidence in her assessment of their relative likelihood.

This example also suggests that mixtures of non-comonotonic acts can be appealing for an individual who might informally be described as "ambiguity-averse". As was just noted, mixtures of f_g and f_b can reduce or eliminate the dependence of the final outcome upon the realization of g rather than b, and hence provide a *hedge against ambiguity*. The DM under consideration finds this appealing: $\frac{1}{2} f_g + \frac{1}{2} f_b \succ f_g \sim f_b$.

Schmeidler [25] suggests that this "preference for mixtures" may be taken as a behavioral definition of *ambiguity aversion*. Formally, say that an individual is ambiguity-averse if, for all $f, g \in \mathcal{F}_0$, $f \succeq g$ implies $\alpha f + (1-\alpha) g \succeq g$. Schmeidler then shows that a CEU individual is ambiguity-averse if and only if the capacity representing her preferences is *convex*: that is, for all events $E, F \in \Sigma$, $v(E \cup F) + v(E \cap F) \geq v(E) + v(F)$. For instance, the capacity in Eq. (2.3) is convex.

2.3. **Multiple Priors and Maxmin Expected Utility.** Gilboa and Schmeidler [16] propose an alternative rationalization of the preferences $f_r \succ f_g$ and $f_{rb} \prec f_{gb}$ in the Ellsberg paradox:

> One conceivable explanation of this phenomenon which we adopt here is as follows: ...the subject has too little information to form a prior. Hence (s)he considers a *set* of priors as possible. Being [ambiguity] averse, s(he) takes into account the *minimal* expected utility (over all priors in the set) while evaluating a bet ([16, p. 142]).[4]

Formally, preferences admit a *maxmin expected utility* (MEU) representation if, given a utility function u and a weak* closed (cf. [1, p. 207]), convex set C of probability charges on S, for all $f, g \in \mathcal{F}_0$, $f \succeq g$ if and only if

$$\min_{P \in C} \int u(f)\, dP \geq \min_{P \in C} \int u(g)\, dP.$$

The modal rankings in the Ellsberg paradox are consistent with MEU, with $u(10) > u(0)$ and, for example,

$$C = \left\{ P \in \Delta(S, \Sigma) \; : \; P(\{r\}) = \frac{1}{3} \right\}. \tag{2.4}$$

Gilboa and Schmeidler's axiomatization of the MEU decision rule features two key axioms: *C-Independence* and Ambiguity Aversion. The latter was stated in the previous subsection; C-Independence requires that, for all acts $f, g \in \mathcal{F}_0$ and all *constant* acts, or lotteries, $p \in \Delta(X)$, $f \succeq g$ if and only if $\alpha f + (1-\alpha) p \succeq \alpha g + (1-\alpha) p$.

[4]For a formal analysis of this interpretation of multiple priors, see Siniscalchi[26].

Thus, relative to the full Independence axiom, preference reversals are ruled out only for mixtures with constant acts.

Intuitively, mixing an act with a constant does not provide any hedging opportunities; rather, such mixtures only change the "scale and location" of an act's utility profile. Thus, the requirement formalized by C-Independence is consistent with the discussion in the preceding subsection; indeed, CEU preferences satisfy C-Independence. On the other hand, MEU preferences may violate Comonotonic Independence (see Klibanoff [19] for an example and further discussion).

Ambiguity-averse CEU preferences satisfy both C-Independence and Ambiguity Aversion (in addition to other structural axioms); thus, they are MEU preferences. Schmeidler [25] shows that, in particular, the convex capacity v representing an ambiguity-averse CEU preference is the *core* of the set C of priors in the MEU representation of the same preferences: that is, $C = \{P \in \Delta(S,\Sigma) : \forall E \in \Sigma, P(E) \geq v(E)\}$. For instance, the capacity v in Eq. (2.3) is the core of the set C in Eq. (2.4).

3. Other Models, Updating, and Dynamic Choice

A generalization of the MEU model, related to Hurwicz' α-maxmin criterion (cf. [21, p. 304]), sometimes appears in applications; given a utility function u, a weak*-closed, convex set C of priors, and a number $\alpha \in [0,1]$, $f \succeq g$ if and only if

$$\alpha \min_{P\in C} \int u(f)\, dP + (1-\alpha) \max_{P\in C} \int u(f)\, dP \geq \alpha \min_{P\in C} \int u(g)\, dP + (1-\alpha) \max_{P\in C} \int u(g)\, dP;$$

thus, MEU corresponds to the case $\alpha = 1$. An axiomatization is provided in Ghirardato, Maccheroni and Marinacci [12].

Truman Bewley [3] proposes an alternative approach to ambiguity. In both the CEU and MEU models, the DM responds to ambiguity by essentially evaluating different acts using different "decision weights". Bewley instead suggests that preferences may simply be *incomplete* in the presence of ambiguity. He axiomatizes the following partial decision rule: for a given utility function u and weak* closed, convex set C of priors, $f \succeq g$ if and only if

$$\forall P \in C, \quad \int u(f)\, dP \geq \int u(g)\, dP.$$

For instance, in Ellsberg's three-color-urn example, if the set C is chosen as above, the DM is unable to rank the acts f_r and f_g, as well as the acts f_{rb} and f_{gb}. Notice that preferences satisfy the Independence axiom in Bewley's model: ambiguity manifests itself solely through incompleteness.

Ambiguity can also be modeled introducing *second-order probabilities*. For instance, Klibanoff, Marinacci and Mukerji [20] axiomatize the following decision rule:

$$\forall f,g \in \mathcal{F}_0, \quad f \succeq g \quad \Leftrightarrow \quad \int_{\Delta(S)} \varphi\left(\int_S u(f)\, dP\right) d\mu \geq \int_{\Delta(S)} \varphi\left(\int_S u(g)\, dP\right) d\mu,$$

where μ is a probability measure over the set $\Delta(S)$ of probability charges on the finite state space S, and φ is a "second-order utility function". A notion of ambiguity aversion is characterized by concavity of φ. See also Ergin and Gul [11].

Recent contributions aim at characterizing ambiguity without restricting attention to specific decision models, and without relying on functional-form considerations. Epstein and Zhang [10] propose a definition of "unambiguous event" that is based solely on preferences. Under suitable structural axioms, preferences over acts that are measurable with respect to such "subjectively unambiguous" events are *probabilistically sophisticated* in the sense of Machina and Schmeidler [22]; this indicates that the proposed behavioral definition characterizes absence of ambiguity. See also Epstein [8] for a related assessment of Schmeidler's definition of ambiguity aversion.

Ghirardato, Maccheroni and Marinacci [12] note that, in models such as CEU and MEU, ambiguity manifests itself via violations of the Anscombe-Aumann Independence axiom. Thus, they propose to deem an act f "unambiguously preferred" to an act g if $\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h$ for all $\alpha \in (0,1]$ and all $h \in \mathcal{F}_0$. They show that unambiguous preference admits a Bewley-style representation, characterized by a set C of priors which is a singleton if and only if the original preference is SEU. In light of this result, they suggest that the DM perceives ambiguity whenever C is not a singleton. See also Ghirardato and Marinacci [14].

To highlight the differences between these definitions, consider a probabilistically sophisticated, non-SEU preference. According to the Epstein-Zhang definition, all events are subjectively unambiguous, whereas the Ghirardato-Maccheroni-Marinacci approach concludes that some ambiguity is perceived.

The modal preferences in the Ellsberg paradox constitute a violation of the *Sure-Thing Principle*, which is arguably the centerpiece of Leonard Savage's axiomatization of SEU [23]; indeed, this was a main focus of Ellsberg's seminal article. However, the Sure-Thing Principle also plays a key role in ensuring that conditional preferences are well-defined and "dynamically consistent"; finally, it provides a foundation for Bayesian updating. Thus, since ambiguity leads to violations of the Sure-Thing Principle, defining updating and ensuring a suitable form of dynamic consistency for MEU, CEU and similar decision model presents some challenges.

Gilboa and Schmeidler [17] axiomatize Dempster-Shafer updating of capacities (cf. Dempster [6]) and "maximum-likelihood updating" of multiple priors for ambiguity-averse CEU preferences. Prior-by-prior updating for MEU preferences is axiomatized in Jaffray [18].

These updating rules may lead to "dynamic inconsistencies", i.e. preference reversals: the ranking of two acts may change after learning that a (typically ambiguous) event has occurred. Epstein and Schneider [9] instead axiomatize a model of recursive MEU preferences by explicitly imposing dynamic consistency with respect to a pre-specified filtration. Note that dynamic consistency imposes some restrictions on the set of MEU priors: see [9] for further discussion. Dynamic choice under ambiguity is currently an area of active research.

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