Foundations for Structural Preferences

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Abstract

Structural rationality (Siniscalchi, 2020) is a notion of optimality that reflects a player's ex-ante perspective in a dynamic game, and formalizes the notion of *robust caution*. The player is cautious in the sense that she takes all possible observable occurrences in the game under consideration, including those she regards implausible ex-ante. However, she does so robustly, without committing to an exogenous ranking of implausible events.

The present paper provides an axiomatic foundation for structural preferences, and argues that this optimality criterion is an effective model of *surprises* in general dynamic decision problems. It also draws connections between robust caution and the notions of justifiability (Lehrer and Teper, 2011), lexicographic expected utility (Blume, Brandenburger, and Dekel, 1991) and Knightian uncertainty (Bewley, 2002).

Keywords: conditional probability systems, lexicographic expected utility, Knightian uncertainty, structural rationality, surprises.

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1 Introduction

This paper provides an axiomatic analysis of individuals' behavior in the face of surprises. It focuses on *structural rationality*, a model of decision-making introduced in Siniscalchi (2020, SRDG henceforth) in the context of dynamic games.

A *surprise* is an event the decision-maker did not expect to happen. With standard expected-utility (EU) preferences, a surprise is a zero-probability event. Since such events are *null* in the sense of Savage (1972), they are behaviorally irrelevant: they literally do not influence choices or plans. Surprise events are also prevalent in dynamic game theory, where they are also modelled as zero-probability events. However, the standard notion of best response in such games, namely sequential rationality (Kreps and Wilson, 1982), *does* allow for surprise events to influence players' plans of action. For instance, in a centipede game, while the player moving first is supposed to end the game with probability one, sequential rationality nevertheless implies that the second player will plan to also end the game, in case the first deviates. Yet this plan will not be implemented, unless the first player unexpectedly deviates.

A fundamental tension within the notion of sequential rationality is that, on one hand, it predicts that players will respond optimally to surprises *once they occur*; on the other hand, it implies that ex-ante choices are consistent with EU, and hence cannot reveal beliefs or plans following surprise events. SRDG illustrates this issue, and shows that structural rationality provides a way to resolve this tension. The reason is that structural rationality reflects a player's *ex-ante* view of the game, but incorporates a notion of *robust caution*. Under structural rationality the player's ex-ante behavior does take all possible surprises into account; however, no exogenous assumption is made as to the relative lihelihood of such surprises, beyond what can be deduced from the dynamic structure of the game.

This paper argues that robust caution provides a compelling foundation for a theory of surprise. I provide an axiomatic characterization of structural rationality that emphasizes how the *possibility* of surprises can shape preferences—and to what extent it does not. In this respect, structural rationality emerges as a *minimal* or *permissive* theory of individuals' response to surprises, precisely because it does not require that the individual evaluate the relative (im)plausibility of *all* unlikely events. In this respect, the proposed theory is the polar opposite of lexicographic EU (Blume et al., 1991), which loosely speaking requires the individual to fully specify one particular (im)plausibility ranking of all unlikely events. The minimality property of structural preferences can also be interpreted as a specific form of attitude toward ambiguity, or more precisely Knightian uncertainty (Bewley, 2002). The following extended example (which will be continued in Section 3.4, after the main definitions are in place) illustrates these points.

Example 1 (R&D competition) Apple is considering a major R&D investment in foldable display technology. At the present time, Apple does not have a marketable product, and is certain that no other competitor does. However, it is aware that either Samsung or Huawei might be able to successfully bring a foldable phone to market at a later date. Furthermore, if Samsung is successful, it may manage to use the same technology in a foldable tablet, which would result in further competition for Apple.

To model this situation, let the state space be $\Omega = \{n, st, sn, h\}$ be the state space; the states indicate, respectively, the events that no competitor develops a successful foldable display, that Samsung develops a foldable phone and, subsequently, a foldable tablet, that Samsung develops a foldable phone but not a tablet, and finally that Huawei develops a foldable phone. Apple can make decisions at four points in time: at the ex-ante stage, denoted ϕ; upon learning Samsung (resp. Huawei) has successfully developed a foldable phone, i.e., conditional upon event $F_s = \{st, sn\}$; (resp. $F_h = \{h\}$); and upon learning that Samsung has developed a foldable tablet, i.e. conditional upon $F_{st} = \{s, t\}$.

Apple holds well-defined beliefs at each deision point: its prior $\mu(\cdot|\Omega)$ assigns probability one to n, so $\mu(n|\Omega) = 1$. In general, Apple's belief conditional upon event $F \in \{\Omega, F_s, F_h, F_{st}\} \equiv \mathscr{F}$ is denoted $\mu(\cdot|F)$. Assume that $\mu(\{sn\}|F_s) = 1$ (that is, Apple remanis pessimistic about

Samsung's ability to develop a foldable tablet, even if they develop a foldable phone). Beliefs conditional on F_{st} and F_h are trivial: if Apple learns that the true state is st (resp. h), it must assign probability one to it.

I focus here on intepreting Apple's beliefs in terms of surprises; Section 3.4 develops this example into an actual decision problem for Apple. First, since $\mu(F_s|\Omega) = \mu(F_h|\Omega) = 0$, both F_s and F_h are surprise events. Since $F_{st} \subset F_s$, a fortiori $\mu(F_{st}|\Omega) = 0$, so F_{st} is also surprising. However, there is a natural sense in which F_{st} is *more surprising* (more implausible) than F_s: by the time Apple has observed F_{st}, it will have been surprised twice. In particular, Apple must first unexpectedly observe F_s, and then, as a further surprise, F_{st}. This conclusion is determined by the structure of the decision tree (specifically, the fact that $F_{st} \subset F_s$) and by Apple's beliefs (specifically, the fact that $\mu(sn|F_s) = 1$, so that F_{st} is surprising even from the perspective of F_s).

Structural preferences rely *solely* on these conclusions. Why does this reflect a parsimonious or minimal apprach to representing beliefs? Consider F_s and F_h. Both events are surprising from the ex-ante perspective. However, just based on Apple's conditional beliefs as defined above, and on the structure of the trees, one cannot rank F_s and F_h in terms of relative plausibility. In fact, even F_{st} and F_h cannot be ranked. Despite the fact that F_{st} represents a "double surprise," one cannot rule out that, in Apple's estimation, it is even less plausible for Huawei to develop a foldable smartphone than for Samsung to develop a foldable tablet. Of course, the opposite could be true. Structural rationality is defined so as not to rely on such non-robust plausibility rankings: see Definition 4.

One can provide an alternative representation of structural preferences that emphasizes this, and draws a connection with both lexicographic EU and Knightian uncertainty. (The details are in Section 3.3.) Recall that a *lexicographic probability system* (LPS) is an ordered list $(p_1,\ldots,p_n)$ of probabilities on Ω; p_1 is interpreted as the most salient belief, and p_n as the least salient. Say that an LPS $(p_1,\ldots,p_n)$ *generates* Apple's beliefs if, conditional upon each event $F \in \mathscr{F}$, Apple's beliefs are the updates of the lowest-index belief that assigns posiive probability to F. For instance, the LPSs

$$p = (p_1, p_2, p_3, p_4) \quad \text{such that} \quad p_1(n) = 1,\ p_2(sn) = 1,\ p_3(st),\ p_4(h) = 1$$

and

$$q = (q_1, q_2, q_3, q_4) \quad \text{such that} \quad q_1(n) = 1,\ q_2(h) = 1,\ q_3(sn),\ q_4(st) = 1$$

both generate Apple's beliefs. However, notice that, according to p, state h only receives positive probability under the least salient belief, whereas under q it is the second most salient state.

It can be shown that, given two acts f, g defined on Ω, f is strictly preferred to g according to structural preferences *if and only if it is lexicographically strictly preferred to it according to* ***all*** *LPSs that generate Apple's beliefs* (see Proposition 2). This is reminiscent of Bewley's decision criterion (Bewley, 2002), except that the representation involves sets of LPSs rather than sets of probabilities.

In turn, this suggests an interpretation that is reminiscent of ambiguity. Apple is able to form probabilistic beliefs at every decision point I. However, they are not willing to commit to specific LPSs that would necessarily incorporate additional information about the relative likelihood of different surprise events (compare the LPSs p and q defined above). Thus, Apple considers *all* such LPSs as possible, and adopts a unanimity criterion.

This paper is organized as follows. Section 2 introduces the notation and setup. Section 3 formalizes structural preferences. Section 4 presents the axiomatic characterization. Finally, Section 5 concludes with a discussion of the related literature, and future work.

2 Setup

I now describe the decision environment faced by players in a dynamic game. This is mainly for homogeneity with the notation in SRDG. However, one can intepret the environment as a single-person decision problem, in which one individual, i, is engaged in a game against "Nature," represented by player $-i$.

Subsection 2.1 introduces dynamic game forms. Subsection 2.2 describes the domain of each player's preferences, which includes the set of Anscombe and Aumann (1963)–style acts that depend upon coplayers' strategies as well as, possibly, additional sources of uncertainty.

2.1 Dynamic game forms

This paper considers dynamic games with imperfect information. The analysis only requires that certain familiar reduced-form objects be defined. (The Online Appendix of SRDG describes how these objects are derived from a complete description of the underlying game.) I omit a specification of payoffs. Instead, the next section indicates how consequences are attached to strategy profiles. Together with a specification of players' utility functions over consequences, this determines payoff assignments as well.

A dynamic game form is represented by a tuple $\big(N,(S_i,\mathscr{I}_i)_{i\in N},S(\cdot)\big)$, where:

- N is the set of **players**.
- S_i is the set of **strategies** of player i.
- $\mathscr{I}_i$ is the collection of **information sets** player i; it is convenient to assume that the **root**, ϕ, is an information set for all players.
- For every $i\in N$ and $I\in\mathscr{I}_i$, $S(I)$ is the set of strategy profiles $(s_j)_{j\in N}\in\prod_j S_j$ that **reach** I. In particular, for every $i\in N$, $S(\phi)=S$.

I adopt the usual conventions for product sets: $S_{-i}=\prod_{j\neq i}S_j$ and $S=S_i\times S_{-i}$. I assume that the game has **perfect recall**, as per Def. 203.3 in OR. In particular, this implies that, for every $i\in N$ and $I\in\mathscr{I}_i$, $S(I)=S_i(I)\times S_{-i}(I)$, where $S_i(I)=\mathrm{proj}_{S_i}S(I)$ and $S_{-i}(I)=\mathrm{proj}_{S_{-i}}S(I)$. If $s_{-i}\in S_{-i}(I)$, say that s_{-i} **allows** I.[1]

[1]That is: if i's coplayers follow the profile s_{-i}, I *can* be reached; whether it *is* reached depends upon whether or not i plays a strategy in $S_i(I)$.

2.2 Choice domain for a distinguished player

For the remainder of this paper, I fix a dynamic game form $\big(N, (S_i, \mathscr{I}_i)_{i\in N}, S(\cdot)\big)$ and focus on one distinguished player $i \in N$. In addition, when it is possible to do so without causing confusion, I will drop the player index i from the notation.

Fix a convex set X of material outcomes; for instance, X may be the set of simple lotteries on some prize space Y, as in Anscombe and Aumann (1963).

The state space for player i comprises her coplayers' strategies, as well as possible concomitant uncertainty. Such uncertainty may represent the unobserved realization of an randomizing device, as in the elicitation game analyzed in SPSR. It may also represent *incomplete information*: for instance, the unobserved, common value of an object being auctioned, or private signals received by coplayers. Finally, it may represent coplayers' *epistemic types*, as e.g. in Battigalli and Siniscalchi (1999b).

Formally, consider a set W of **concomitant uncertainty**, endowed with a sigma-algebra $\mathscr{W}$. The domain of player i's overall uncertainty is $\Omega = S_{-i} \times W$, endowed with the product sigma-algebra $\Sigma = 2^{S_{-i}} \times \mathscr{W}$.

The distinguished player i is characterized by a preference relation $\succ$ on the set of **acts** $f : \Omega \to X$, denoted $\mathscr{A}$.

In SRDG, attention is confined to acts associated with a strategy $s_i \in S_i$. Fix an *outcome function* $\xi_i : S \times W \to X$; the interpretation is that, if strategy profile $s \in S$ is reached, and the realization of player i's concomitant uncertainty is $w \in W$, then player i's material outcome is $\xi_i(s, w)$. Also consider a strategy s_i of i. Then, for every state $\omega = (s_{-i}, w)$, the profile (s_i, s_{-i}) and the realization w of concomitant uncertainty lead to outcome $\xi_i((s_i, s_{-i}), w)$. This determines an act $f^{s_i} : \Omega \to X$. Thus, the player's preference $\succ$ over $\mathscr{A}$ also induces a preference over strategies. However, the axiomatic analysis in the present paper considers arbitrary acts, not just those associated with strategies.

The class of **conditioning events** for player i is defined by

$$\mathscr{F} = \{\Omega\} \cup \{S_{-i}(I) \times W \,:\, I \in \mathscr{I}_i\}. \tag{1}$$

Observe that Ω is always a conditioning event, even if there is no information set $I \in \mathscr{I}_i$ such that $S_{-i}(I) \times W = \Omega$. This is convenient (though not essential) to relate Structural preferences to ex-ante expected-payoff maximization.

Finally, denote by $B_0(\Sigma)$ the set of Σ-measurable real functions with finite range[2], and by $B(\Sigma)$ its sup-norm closure.

3 Conditional Beliefs and Structural Preferences

I now introduce structural preferences. The definitions below adapt those in SRDG to the more general choice domain considered here.

3.1 Consistent Conditional Probability Systems

Following Ben-Porath (1997); Battigalli and Siniscalchi (1999a, 2002), Myerson (1986) or Kohlberg and Reny (1997), player i's beliefs are represented using conditional probability systems (Rényi, 1955); however, I impose a more stringent consistency condition on beliefs across different conditioning events. For a measurable space $(Y, \mathscr{Y})$, $\Delta(\mathscr{Y})$ denotes the set of *finitely* additive probability measures on $\mathscr{Y}$.

Definition 1 *A **consistent conditional probability system (CCPS)** for player* i *is an array* $\mu = \left(\mu(\cdot|F)\right)_{F\in\mathscr{F}} \in \Delta(\Sigma)^{\mathscr{F}}$ *such that*

(1) for all $F \in \mathscr{F}$, $\mu(F|F) = 1$*;*

[2]Recall that, while S_{-i} is assumed to be finite, W is not.

(2) for every $F_1, \ldots, F_L \in \mathscr{F}$ and $E \subseteq F_1 \cap F_L$,

$$\mu(E|F_1) \cdot \prod_{\ell=1}^{L-1} \mu(F_\ell \cap F_{\ell+1}|F_{\ell+1}] = \mu(E|F_L) \cdot \prod_{\ell=1}^{L-1} \mu(F_\ell \cap F_{\ell+1}|F_\ell) \tag{2}$$

Denote the set of consistent CPSs for player i by $\Delta(\Sigma, \mathscr{F})$.

Take $L = 2$ in property (2), and assume that $F_1 \subseteq F_2$. Then Eq. (2) reduces to

$$\mu(E|F) \cdot \mu(F_1)|F_2) = \mu(E|F_2), \tag{3}$$

which, together with property (1), characterizes "conditional probability systems" with conditioning events $\mathscr{F}$, as defined in Rényi (1955). Eq. (3) can be interpreted from an *interim* perspective as requiring that player i update her beliefs in the usual way whenever possible: if $E \subseteq F_1$ and $\mu(F_1|F_2) > 0$, then $\mu(E|F_1) = \frac{\mu(E|F_2)}{\mu(F_1|F_2)}$. Eq. (2) imposes additional restrictions, which are easiest to interpret from an *ex-ante* perspective. Suppose that $\mu(\cdot|F_1), \ldots, \mu(\cdot|F_L)$ are the updates of some $P \in \Delta(\Sigma)$ that assigns positive probability to each F_ℓ, $\ell = 1, \ldots, L$. Then Eq. (2) reduces to

$$\frac{P(E)}{P(F_1)} \cdot \prod_{\ell=1}^{L-1} \frac{P(F_\ell \cap F_{\ell+1})}{P(F_{\ell+1})} = \frac{P(E)}{P(F_L)} \cdot \prod_{\ell=1}^{L-1} \frac{P(F_\ell \cap F_{\ell+1})}{P(F_\ell)}, \tag{4}$$

which holds because the quantities in the numerators and denominators of either side are the same—they just appear in different order. Intuitively, Eq. (2) requires that the same condition hold even if the prior probability of one or more conditioning event F_ℓ is "infinitesimal." To put it differently, under Eq. (2), one can think of the array μ as being derived by updating a prior "probability measure" on Ω that assigns infinitesimal weight to one or more conditioning events.

3.2 Structural Preferences

Differently from SRDG, I first define structural preferences in terms of CCPSs. I then provide a characterization in terms of lexicographic probability systems (see SRDG, Section 7.G). While

SRDG only defines preferences over strategies, this section defines them over the set $\mathscr{A}$ of acts. Througout this section, fix (i) a CCPS μ, and (ii) an affine utility function $u : X \to \mathbb{R}$, for the designated player i.

The CCPS μ for player i induces a preorder over conditioning events, as follows.

Definition 2 (cf. SRDG, Definition 5) *For all $D, E \in \mathscr{F}$, $D \geq^{\mu} E$ if there are $F_1, \ldots, F_L \in \mathscr{F}$ such that $F_1 = E$, $F_L = D$, and for all $\ell = 1, \ldots, L-1$, $\mu(F_{\ell+1}|F_\ell) > 0$.*

SRDG shows that the player's CCPS pins down a unique probability measure for each equivalence class of $\geq^{\mu}$:

Proposition 1 (cf. SRDG, Proposition 3) *Fix $F \in \mathscr{F}$. There is a unique $P \in \Delta(\Sigma)$ such that $P(\cup\{G : F =^{\mu} G\}) = 1$ and, for all $G \in \mathscr{F}$ with $F =^{\mu} G$, $P(G) > 0$ and $\mu(\cdot|G) = P(\cdot|G)$.*

Proposition 1 ensures that the following definition is well-posed:

Definition 3 (SRDG, Definition 6) *For every $F \in \mathscr{F}$, let $P_\mu(F)$ be the unique probability identified in Proposition 1.*

In particular, $P_\mu(\Omega) = \mu(\cdot|\Omega)$. Finally, structural preferences over acts are defined as follows.

Definition 4 (cf. SRDG, Theorem 1) *For every pair of acts $f, g \in \mathscr{A}$, f is (strictly)* ***structurally preferred*** *to g given (u, μ) ($s_i \succ^{u,\mu} t_i$) iff there are $F_1, \ldots, F_M \in \mathscr{F}$ with $\int u \circ f \, dP_\mu(F_m) > \int u \circ g \, dP_\mu(F_m))$ for $m = 1, \ldots, M$, and $u(f(\omega)) \geq u(g(\omega))$ for $\omega \notin \bigcup_{m=1}^{M} \bigcup_{G \in \mathscr{F} : F_m \geq^{\mu} G} G$.*

3.3 Characterization via LPSs

I conclude by stating an equivalent characterization of structural preferences that highlights the connection with lexicographic EU and Knightian uncertainty. See SRDG, Section 7.G.

A **lexicographic probability system (LPS)** is an ordered list $p = (p_1, \ldots, p_n) \in \Delta^n(\Omega)$, $n \geq 1$. An LPS $(p_1, \ldots, p_n)$ **generates** the CCPS $\mu \in \Delta(\Sigma, \mathscr{F})$ if, for every $F \in \mathscr{F}$, there is $\ell \in \{1, \ldots, n\}$

with $p_m(F) = 0$ for all $m = 1, \ldots, \ell - 1$, $p_\ell(F) > 0$, and $\mu(\cdot|F) = p_\ell(\cdot|F)$; that is, p_ℓ is the lowest-index belief that assigns positive probability to F, and by conditioning on F one obtains $\mu(\cdot|F)$. Finally, given acts $f, g \in \mathscr{A}$ and a function $u : X \to \mathbb{R}$, say that f is **lexicographically strictly preferred** to g given the LPS $p = (p_1, \ldots, p_n)$, written $f \succ^{u,p} g$, if the vector $\left(\int u \circ f \, dp_\ell\right)_{\ell=1}^n$ is lexicographically strictly greater than the vector $\left(\int u \circ g \, dp_\ell\right)_{\ell=1}^n$.

Proposition 2 (SRDG, Theorem 5) *Fix a CCPS μ and a non-costant, affine $u : X \to \mathbb{R}$. For all acts $f, g \in \mathscr{A}$, $f \succ^{u,\mu} g$ if and only if $f \succ^{u,p} g$ for all LPSs p that generate μ.*

3.4 Back to Example 1

In the example described in the Introduction, Nature's strategy space is $S_{-i} = \{n, st, sn, h\}$, and this is also the relevant overall state space for Apple: $\Omega = S_{-i}$. The conditioning events are $\mathscr{F} = \{\Omega, F_s, F_{st}, F_h\}$, where $F_s = \{st, sn\}$, $F_{st} = \{st\}$, and $F_h = \{h\}$.. Apple's beliefs are represented by the array $\mu \in \Delta(\Omega)^{\mathscr{F}}$ of probabililities given by $\mu(\{n\}|\Omega) = 1$, $\mu(\{sn\}|\{st, sn\}) = 1$, and $\mu(\{st\}|\{st\}) = \mu(\{h\}|\{h\}) = 1$. One can verify that this is indeed a CCPS. Moreover, one has $\Omega >^\mu \{sn, st\} >^\mu \{st\}$ and $\Omega >^\mu \{h\}$. Thus, $P_\mu(F) = \mu(\cdot|F)$ for all $F \in \mathscr{F}$.

Consider the following three strategies that Apple might adopt. The first is $f =$ "do nothing." This entails no investment, but a potential loss of market share in case one of Apple's competitors successfully develops foldable-screen technology; the loss is especially large if Samsung is able to develop foldable tablets. The second is $g =$ "invest in foldable screens only if Samsung develops a foldable phone." In this case, Apple still cannot compete with Samsung on foldable smartphones, and in addition it bears development costs; however, it effectively competes with Samsung on foldable tablets, which partially offsets its losses in the smartphone market. The third is $k =$ "invest in foldable screens no matter what." This entails an upfront investment, but it fends off Samsung's challenge in case it develops foldable screens, and actually improves Apple's position relative to Huawei in case the latter develops a foldable smartphone. The payoffs are described in Table I.

s_i	n	sn	st	h
f	0	-2	-5	-1
g	0	-3	-2	$-1+x$
k	-1	0	0	1

Table I: Investing in foldable screens

Assume that utility is linear. Applying Definition 4, one immediately gets $f \succ^{u,\mu} h$ and $g \succ^{u,\mu} k$. Despite the fact that, if h occurs, k does better than both f and g, the ex-ante development cost trumps this consideration.

How about f vs. g? Assume first that $x = 0$. Then, Apple's pessimism on Samsung's ability to develop foldable tablets even if it succeeds in developing foldable phones is the defining factor. Take $M = 1$ and $F_1 = \{st, sn\}$: then f yields an expected payoff of -2 given $P_\mu(F_1) = \mu(\cdot|F_1)$, which is concentrated on sn; g instead yields an expected payoff of -3. Furthermore for all $\omega \notin F_1$ (note that $\{st\} \subset F_1$, so there is no need to consider it separately), f and g yield the same expected payoff. Hence, $f \succ^{u,\mu} g$.

Suppose instead that $x > 0$. Then g yields a higher expected payoff in state h than f. Note that state h does not belong to any $F \in \mathscr{F}$ with $F_1 \succcurlyeq^\mu F$. Thus, the inequality $f(h) < g(h)$ is inconsistent with $f \succ^{u,\mu} g$, per Definition 4. How about the opposite strict ranking? The only event conditional upon which g does better than f is $F_1 = \{h\}$; however, since $f(sn) > g(sn)$ and state sn does nto belong to any $F \in \mathscr{F}$ with $F_1 \geq^\mu F$, it is also not the case that $g \succ^{u,\mu} f$. Thus, f and g are *not ranked*.

In this case, the fact that Apple is unable to compare the relative likelihood of the surprise events $\{st, sn\}$ and $\{h\}$ prevents a comparison of the policies f and g. Thus, structural preferences are incomplete; however, they are transitive (this is easiest to see from the characterization in Proposition 2) and so admit maximal elements in finite choice sets.

4 Behavioral analysis

The starting point for the axiomatization of structural preferences is Theorem 13.3 in Fishburn (1970) (see also Kreps, 1988, Chapter 7). I reproduce this result for ease of reference. Recall that $\succ$ is **irreflexive** if not $f \succ f$; it is **asymmetric** if $f \succ g$ implies not $g \succ f$; and **negatively transitive** if, for all $f, g, h \in \mathscr{A}$, not $f \succ g$ and not $g \succ h$ imply not $f \succ h$. An asymmetric, negatively transitive preference is irreflexive and transitive, and an irreflexive, transitive relation is asymmetric (cf. Fishburn, 1970, §2). Also, event $E \in \Sigma$ is **null** (Savage, 1972) if, for all $f, g \in \mathscr{A}$ such that $f(\omega) = g(\omega)$ for all $\omega \notin E$, neither $f \succ g$ nor $g \succ f$ hold.

Theorem 1 (Fishburn, 1970) *Suppose that the following axioms hold for all $f, g, h \in \mathscr{A}$:*[3]

Axiom B1. *$\succ$ is irreflexive and negatively transitive;*

Axiom B2. *$f \succ g$ and $\alpha \in (0,1)$ imply $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$;*

Axiom B3. *$f \succ g$ and $g \succ h$ imply that $\alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$ for some $\alpha, \beta \in (0,1)$;*

Axiom B4. *$x \succ y$ for some $x, y \in X$;*

Axiom B5. *If $E \in \Sigma$ is not null, then for all $x, y \in X$, $xEf \succ yEf$ iff $x \succ y$.*

Then there are a non-constant $u : X \to \mathbb{R}$ and $p \in \Delta(\Sigma)$ such that $f \succ g$ if and only if $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g$. Furthermore, u is unique up to positive linear transformations, and p is unique.

Axxiom B1 states that $\succ$ is a weak order (using Fishburn's terminology). B2 is the Independence axiom, and B3 is Archimedean continuity. B4 is non-degeneracy. Finally, B5 is the state-independence or monotonicity axiom. Structural preferences essentially restrict negative transitivity and Archimedean continuity to preferences over prizes, as well as preferenes

[3] I restate Fishburn's result using the present notation. Also, Fishburn considers an additional axiom, but this is only required to accommodate non-simple acts.

over arbitrary acts conditional on the events in $\mathscr{F}$. Furthermore, structural preferences maintain Independence and a suitable version of non-degeneracy, and relax negative transitivity to transitivity for the prior preference $\succ$. This in turn requires tightening state-independence slightly.

Axiom 1 (Strict Partial Order) *$\succ$ is irreflexive and transitive.*

Axiom 2 (Negative Transitivity for Prizes) *For all $x, y, z \in X$, not $y \succ x$ and not $z \succ y$ imply not $z \succ x$.*

Axiom 3 (Independence) *For all $f, g, k \in \mathscr{A}$, $f \succ g$ if and only if $\alpha f + (1-\alpha)k \succ \alpha g + (1-\alpha)k$*

Axiom 4 (Monotonicity) *For all $x, y \in X$, $E \in \Sigma$, and $f, g \in \mathscr{A}$: if not $y \succ x$, then $yEf \succ g$ implies $xEf \succ g$ and $g \succ xEf$ implies $g \succ yEf$.*

The fact that Axiom 4 implies Fishburn's B5 is established in Observation 2 in the Appendix.

As in the case of atemporal expected-utility preferences, Axiom 3 implies Savage's Sure-Thing Principle (Postulate P2).

Remark 1 *Assume Axiom 3. For all $f, g, k, k' \in \mathscr{A}$, and all $E \in \Sigma$: $fEk \succ gEk$ if and only if $fEk' \succ gEk'$.*

Hence, one can define *conditional preferences* following Savage (1972), by modifying pairs so that their act components coincide outside of the conditioning event:

Definition 5 *For all event $E \in \Sigma$, player i's **conditional preference** $\succ_E$ **given** E is defined as follows: for every f, g, $f \succ_E g$ if, for some $k \in \mathscr{A}$, $fEk \succ gEk$.*

Remark 1 ensures that any $k \in \mathscr{A}$ will yield the same conditional ranking of the strategies f, g. Note that $\succ_\Omega$ is simply the prior preference $\succ$.

The remaining axioms involve conditional preferences, as per Def. 5. I emphasize that even these axioms should be interpreted as assumptions on the *prior* preference $\succ$; conditional preferences are solely a convenient way to formalize them.

Axiom 5 (Nondegeneracy) *For all $F \in \mathscr{F}$, there exist $x, y \in X$ such that $x \succ_F y$.*

Axiom 6 (Prize Continuity) *For all $F \in \mathscr{F}$ and $x, y, z \in X$: if $x \succ_F y$ and $y \succ_F z$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha x + (1-\alpha)z \succ_F y$ and $y \succ_F \beta x + (1-\beta)z$.*

Prize continuity is in the spirit of Blume et al. (1991), except that the conditioning events correspond to information sets in the game. Conditional non-degeneracy is a substantive requirement: it implies that all conditioning events "matter" for preferences (even though some may be assigned zero ex-ante probability in the sequential-preference representation).

The final three axioms involve a novel notion of "unlikely" events. As a starting point, recall that, by definition, a *null* event E is behaviorally irrelevant: no matter what prizes two acts f, g deliver at states in E, if $f(\omega) = g(\omega)$ at all states in the complement of E, then f and g are not ranked by $\succ$. With EU preferences, an event is null if and only if it has probability zero.

The proposed notion does allow for prizes delivered at states in E to matter for the ranking of f and g, but only if these acts are not already strictly ranked by dominance on the complement of E. More generally, under the preceding axioms, prizes delivered on E may only help "break ties" that result from considering prizes delivered at states not in E (see e.g. Lemma 16 in the Appendix). Under minimal structural assumptions (implied by the preceding axioms), a null event is negligible, but the converse is not true.

[**Note:** there will be an example here]

Definition 6 *Fix an event $E \in \Sigma$. An event $N \in \Sigma$ is **negligible** given E if, for all $f, g \in \mathscr{A}$, $f(\omega) \succ_E g(\omega)$ for all $\omega \notin N$ implies $f \succ_E g$.*

Under the preceding axioms, the union of two negligible events is negligible (cf. Lemma 11 part 3 in the Appendix).

To stremline the statement of the next two axioms, and to aid interpretation, I introduce the notion of *strategic support* of an event. If there is no additional uncertainty (i.e., if $\Omega = S_{-i}$), the strategic support of E is simply the collection of its non-negligible elements. In the general case ($\Omega = S_{-i} \times W_{-i}$) it is the union of the non-negligible cylinder sets $\{s_{-i}\} \times W_{-i}$.

Definition 7 *The **(strategic) support** of an event $E \in \Sigma$ is $\sigma(E) = \bigcup \big\{ \{s_{-i}\} \times W : \{s_{-i}\} \times W$ is not negligible given $E \big\}$.*

Together with the preceding axioms, the next two axioms ensure that preferences conditional on the strategic support of a conditioning event in $\mathscr{F}$ satisfy the axioms in Theorem 1, and hence identify unique conditional probabilities. The intuition is that, by restricting attention to the strategic support of an event $F \in \mathscr{F}$, one removes the possibility that ties might be broken by considering negligible events in F. Thus, assuming EU behavior on the strategic support of F formalizes the assumption that the *only* departure from expected utility allowed by structural preferences involves the consideration of negligible events to break ties.

Axiom 7 (Non-Negligible Negative Transitivity) *For all $F \in \mathscr{F}$, $\succ_{\sigma(F)}$ is negatively transitive.*

Axiom 8 (Non-Negligible Continuity) *For all $F \in \mathscr{F}$, if $e \succ_{\sigma(F)} f$ and $f \succ_{\sigma(F)} g$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha e + (1-\alpha) g \succ_F f$ and $f \succ_{\sigma(F)} \beta e + (1-\beta) g$.*

Axioms 1–8 are not enough to fully characterize the structural-preference representation. However, they do uniquely identify the individual's utility u and conditional beliefs μ. The first main result of this paper shows that the structural preference $\succ^{u,\mu}$ is in fact the *minimal* relation that satisfies the preceding axioms and identifies u and μ:

Theorem 2 *For any non-constant, affine function $u : X \to \mathbb{R}$ and CCPS $\mu \in \Delta(\Sigma, \mathscr{F})$, $\succ^{u,\mu}$ satisfies Axioms 1–8.*

Furthermore, if $\succ$ satisfies Axioms 1–8, then there exists a non-constant, affine function $u : X \to \mathbb{R}$ unique up to positive affine transformations, and a unique CCPS μ such that, for *every* $f,g\in\mathscr{A}$, $f\succ^{u,\mu} g\Longrightarrow f\succ g$.

One implication is that any preference relation that satisfies Axioms 1–8, but is not structural, evaluates acts through features not reflected in the individual's CPS μ, and in the conditional expectations of acts at each $F\in\mathscr{F}$. [**Note:** examples to come]

The final axiom excludes such extraneous considerations from the evalation of acts. It is useful to consider EU preferences as a starting point. It may well be the case that $f\succ g$, but there is some positive-probability event E such that $f\prec_E g$. Of course, in this case f must do strictly better than g conditional upon $\Omega\setminus E$, so as to "compensate" for the fact that $f\prec_E g$. The same holds for structural preferences, if E has positive prior probability. However, the intuition that structural preferences take surprises into account suggests that some form of compensation should be allowed in case E has zero prior probability. Axiom 9 characterizes the form this compensation can take.

The axiom imposes two key requirements. First, the "compensating event" F such that $f\succ_F g$ must be the union of conditioning events in $\mathscr{F}$, one of which must intersect E. Second, the event F should be at least as "plausible" as E. To impose this plausibility restriction (without recourse to probabilities), I introduce the notion of a *full sequence*:

Definition 8 *A **full sequence** is an ordered list* $F_1,\dots,F_L\in\mathscr{F}$ *such that (i) for every* $\ell=1,\dots,L-1$, $F_{\ell+1}$ *is not negligible given* F_ℓ *and (ii) for every* $\ell=1,\dots,L$*, if* $G\in\mathscr{F}$ *is not negligible given* F_ℓ *and* F_ℓ *is not negligible given* G*, then* $G\in\{F_1,\dots,F_L\}$.

To unpack this definition, observe first that, if an event G is non-negligible given G', then it "matters" for $\succ_{G'}$ just like the strategic support of G'. In this sense, say that G is at least as plausible than G'. Thus, the first part of the definition says that the events $F_1,\dots,F_L$ are ordered in terms of plausibility. Part (ii) imposes a closure condition: if some $G\in\mathscr{F}$ is just as plausible as some F_ℓ, then it also belongs to the full sequence.

Axiom 9 below requires that, if $f\succ g$, the compensating event must be the union of a full sequence of conditioning events, all of which are at least as likely as the event E where f does worse than g. This has two implications, which correspond to the two parts of Definition 8. First, the *only* notion of "at least as plausible" that is allowed to compensate for $f \prec_E g$ is that implied by considering chains of non-negligible events. By definition, this notion is fully determined by preferences at conditional events. Second, the closure requirement rules out the possibility that $f \succ_F g$ on some event F that is at least as plausible than E, but $f \prec_G g$ on some event $G \supset F$ that is just as plausible as F. (Loosely speaking, in this case the compensation might be considered "voided" or "undone" by the fact that $f \prec_G g$.)

Axiom 9 (Sequential compensation) *For all $f, g \in \mathscr{A}$, and all $E \in \Sigma$ with $f(\omega) \prec g(\omega)$ for all $\omega \in E$: if $f \succ g$, then there exist a full sequence $F_1, \ldots, F_L \in \mathscr{F}$ and $x, y \in X$ such that $E \cap F_1 \neq \emptyset$ if $E \neq \emptyset$, and $f \succ_{\cup_\ell F_\ell} x \succ_{\cup_\ell F_\ell} y \succ_{\cup_\ell F_\ell} g$.*

The second main result of this paper can now be stated.

Theorem 3 *Let $\succ$ be a preference on $\mathscr{A}$. The following statements are equivalent:*

1. *$\succ$ satisfies Axioms 1–9;*
2. *there is a non-constant, affine function $u : X \to \mathbb{R}$, and a CCPS μ such that, for all $f, g \in \mathscr{A}$, $f \succ g$ if and only if $f \succ^{u,\mu} g$.*

Furthermore, in (2), u is unique up to positive affine transformations, and μ is unique.

5 Discussion and conclusions

[**Note:** To be written]

A Main result: Preliminaries

This section contains definitions and results from SPDG that will be invoked in the proofs of both necessity and sufficiency. Some of these are from SPDG. The results only concern CPSs and the associated plausibility relations, not preferences.

An ordered list $F_1,\dots,F_L \in \mathscr{F}$ is a **μ-sequence** if $\mu(F_{\ell+1}|F_\ell) > 0$ for all $\ell = 1,\dots,L-1$. A μ-sequence $F_1,\dots,F_L$ is **full** if, for every $\ell = 1,\dots,L$, and $G \in \mathscr{F}$, $\mu(G|F_\ell) > 0$ and $\mu(F_\ell|G) > 0$ imply $G \in \{F_1,\dots,F_L\}$.

The following Remark lists immediate consequences of the definition of μ-sequence. This result applies to arbitrary arrays $(\mu(\cdot|F))_{F\in\mathscr{F}} \in \Delta(\Sigma)^{\mathscr{F}}$, whether or not they are CCPSs.

Remark 2

1. *If $F_1,\dots,F_L$ is a μ-sequence and $1 \le \ell \le m \le L$, then $F_\ell,\dots,F_m$ is a μ-sequence.*

2. *If $F_1,\dots,F_L$ and $G_1,\dots,G_M$ are μ-sequences, and $\mu(G_1|F_L) > 0$, then $F_1,\dots,F_L,G_1,\dots,G_M$ is a μ-sequence.*

3. *$F \ge^\mu G$ iff there is a μ-sequence $F_1,\dots,F_L$ such that $F_1 = G$ and $F_L = F$.*

4. *if $F_1,\dots,F_L$ is a μ-sequence and $1 \le \ell \le m \le L$, then $F_m \ge^\mu F_\ell$.*

5. *if $\mathscr{C}$ is an equivalence class of $\ge^\mu$, then for any $F \in \mathscr{C}$ there is a μ-sequence $G_1,\dots,G_M$ such that $\mathscr{C} = \{G_1,\dots,G_M\}$ and $G_1 = G_M = F$.*

Proof: (1) and (2) are immediate; (3) restates the definition of $\ge^\mu$, and (4) follows from (1) and (3). For (5), let $\{F_1,\dots,F_L\}$ be an enumeration of $\mathscr{C}$ with $F_L = F$. Since in particular $F_L \ge^\mu F_1 \ge^\mu F_2 \ge^\mu \dots \ge^\mu F_L$, for every $\ell = L,\dots,2$ there is a μ-sequence $F^\ell_1,\dots,F^\ell_{L(\ell)}$ with $F^\ell_1 = F_\ell$ and $F^\ell_{L(\ell)} = F_{\ell-1}$. Furthermore, there is a μ-sequence $F^1_1,\dots,F^1_{L(1)}$ such that $F^1_1 = F_1$ and $F^1_{L(1)} = F_L$. Since $F^\ell_{L(\ell)} = F^{\ell-1}_1$ for all $\ell = L,\dots,2$, repeated applications of part (3) shows that

$$F^L_1,\dots,F^L_{L(L)},F^{L-1}_1,\dots,F^2_{L(2)},F^1_1,\dots,F^1_{L(1)}$$

is a μ-sequence that contains $\{F_1,\dots,F_L\}$. Furthermore, $F^L_1 = F_L = F^1_{L(1)}$, so for every $\ell = 1,\dots,L$ and $m = 1,\dots,L(\ell)$, $F^\ell_m \ge^\mu F_L$ and $F_L \ge^\mu F^\ell_m$: that is, $F^\ell_m \in \{F_1,\dots,F_L\}$. Hence, the above displayed equation provides the required μ-sequence $G_1,\dots,G_M$, with $G_1 = G_M = F_L = F$. ■

Lemma 1 (cf. Lemma 2 in SRDG) *Fix $G \in \mathscr{F}$ and consider a μ-sequence $F_1,\dots,F_L \in S_{-i}(\mathscr{I}_i)$ such that $P_\mu(G)(\cup_\ell F_\ell) > 0$. Then there are $\hat{\ell} \in \{1,\dots,L\}$ and $\hat{G} \in \mathscr{F}$ such that $\hat{G} =^\mu G$ and $\mu(F_{\hat{\ell}}|\hat{G}) > 0$. Therefore $F_\ell \geq^\mu G$ for all $\ell = \hat{\ell},\dots,L$.*

Notation: for $F \in \mathscr{F}$, let $B_\mu(F) = \cup\{G \in \mathscr{F} : F =^\mu G\}$. Viewing $B_\mu(\cdot)$ as a function on $\mathscr{F}$, I also denote its range by $\mathscr{B}_\mu(\mathscr{F})$.

Corollary 1 *For all $E,F \in \mathscr{F}$:*

1. *$P_\mu(F)(B_\mu(E)) > 0$ implies $E \geq^\mu F$.*
2. *if $E >^\mu F$, then $P_\mu(E)(F) = 0$.*

Proof: By Remark 2, there is a μ-sequence $F_1,\dots,F_L$ such that $\{F_1,\dots,F_L\}$ is the equivalence class containing E. The Lemma yields $\hat{\ell}$ such that $F_{\hat{\ell}} \geq^\mu F$, and transitivity yields claim 1. For claim 2, if $P_\mu(E)(F) > 0$ then $F \geq^\mu E$, hence not $E >^\mu F$. ■

Corollary 2 *If $F_1,\dots,F_L$ and $G_1,\dots,G_M$ are μ-sequences with $\cup_\ell F_\ell = \cup_m G_m$, then $F_L =^\mu G_M$.*

Proof: $B_\mu(G_M) = \cup_m G_m = \cup_\ell F_\ell$, so $P_\mu(G_M)(\cup_\ell F_\ell) = 1$ and the Lemma yields $\hat{\ell}$ with $F_{\hat{\ell}} \geq^\mu G_M$. By Remark 2 and transitivity, $F_L \geq^\mu G_M$. Similarly, since $P_\mu(F_L)(\cup_m G_m) = 1$, the Lemma yields $\hat{m}$ with $G_{\hat{m}} \geq^\mu F_L$, and again Remark 2 and transitivity imply $G_M \geq^\mu F_L$. ■

Now define

$$\mathscr{F}_\mu = \{\cup_\ell F_\ell : F_1,\dots,F_L \text{ is a } \mu\text{-sequence }\}. \tag{5}$$

and an array $\rho = \big(\rho(\cdot|F)\big)_{F \in \mathscr{F}_\mu} \in \Delta(\Sigma)^{\mathscr{F}_\mu}$ by letting, for every μ-sequence $F_1,\dots,F_L$,

$$\rho(E|\cup_\ell F_\ell) = \frac{P_\mu(F_L)E \cap [\cup_\ell F_\ell]\big)}{P_\mu(F_L)\big(\cup_\ell F_\ell\big)}. \tag{6}$$

By Corollary 2, if $\cup_m G_m = \cup_\ell F_\ell$ for another μ-sequence $G_1,\dots,G_M$, then $G_M =^\mu F_L$ and so $B_\mu(G_M) = B_\mu(F_L)$. Furthermore, by Lemma 1, $P_\mu(F_L)(\cup_\ell F_\ell) \geq P_\mu(F_L)(F_L) > 0$. Thus, the above definition is well-posed.

Note that $\mathscr{F} \subset \mathscr{F}_\mu$; moreover, Remark 2 part 5 implies that also $B_\mu(\mathscr{F}) \subset \mathscr{F}_\mu$. Recall from Section 3.1 that a *conditional probability system* (CPS) is an array of probabilities that satisfies condition (1) in Definition 1 and Eq. (3), though not necessarily Eq. (2). Denote the set of CPSs on a measurable space $(Y, \mathscr{Y})$ with conditioning events $\mathscr{G} \subseteq \mathscr{Y}$ by $\tilde{\Delta}(\mathscr{Y}, \mathscr{G})$. Then ρ is CPS on Ω–though not necessarily a CCPS—that extends μ and is consistent with $P_\mu(\cdot)$.

Lemma 2 *For every $F \in \mathscr{F}$, $\rho(\cdot|F) = \mu(\cdot|F)$ and $\rho(\cdot|B_\mu(F)) = P_\mu(F)$. Furthermore, $\rho \in \tilde{\Delta}(\Sigma, \mathscr{F}_\mu)$.*

Proof: If $F \in \mathscr{F}$, then $\rho(E|B_\mu(F)) = \frac{P_\mu(F)(E\cap F)}{P_\mu(F)(F)} = \mu(E \cap F|F)$ by the chain rule and the fact that ν extends μ. If $F \in B_\mu(\mathscr{F})$, then there is a μ-sequence $F_1, \ldots, F_L$ such that $F = \cup_\ell F_\ell = B_\mu(F_L)$, and since $P_\mu(F_L)(B_\mu(F_L)) = 1$, $\rho(E|F) = P_\mu(F_L)(E \cap [\cup_\ell F_\ell]) = P_\mu(F_L)(E \cap B_\mu(F_L))) = P_\mu(F_L)(E)$.

It remains to be shown that ρ is a CPS. It is immediate that $\rho(F|F) = 1$ for $F \in \mathscr{F}_\mu$. Thus, suppose $E \subseteq F \subseteq G$ for $E \in \Sigma$ and $F, G \in \mathscr{F}_\mu$. Let $F_1, \ldots, F_L$ and $G_1, \ldots, G_M$ be μ-sequences such that $F = \cup_\ell F_\ell$ and $G = \cup_m G_m$. If $\rho(F|G) = 0$ there is nothing to show. Thus, assume $\rho(F|G) > 0$. By definition, this implies that $P_\mu(G_M)(\cup_\ell F_\ell) > 0$, so $F_L \geq^\mu G_M$ by Lemma 1. Furthermore, since $F_L \subseteq \cup_\ell F_\ell \subseteq \cup_m G_m$, $1 = \mu(\cup_m G_m|F_L) \leq \sum_m \mu(G_m \cap F_L|F_L)$, so there is $\hat{m}$ with $\mu(G_{\hat{m}}|F_L) > 0$ and so $G_{\hat{m}} \geq^\mu F_L$; Remark 2 and transitivity then yield $G_M \geq^\mu F_L$.

Thus, $G_M =^\mu F_L$, so $B_\mu(F_L) = B_\mu(G_M)$ and therefore, since $E \subseteq \cup_\ell F_\ell \subseteq G_m$,

$$\rho(E|\cup_\ell G_m) = \frac{P_\mu(G_M)(E)}{P_\mu(G_M)(\cup_\ell G_m)} = \frac{P_\mu(F_L)(E)}{P_\mu(F_L)(\cup_\ell F_\ell)} \cdot \frac{P_\mu(G_M)(\cup_\ell F_\ell)}{P_\mu(G_M)(\cup_m G_m)} = \rho(E|\cup_\ell F_\ell) \cdot \rho(\cup_\ell F_\ell|\cup_m G_m),$$

i.e., Eq. (3) holds. ■

Lemma 3 *Let $F_1, \ldots, F_L \in \mathscr{F}$ be a μ-sequence. Let $m = \min\{\bar{\ell} \in \{1, \ldots, L\} : \forall \ell = \bar{\ell}, \ldots, L-1,\ \mu(F_\ell|F_{\ell+1}) > 0\}$.*

1. *For every $\ell = 1, \ldots, L$, $F_\ell =^\mu F_L$ and $P_\mu(F_L)(F_\ell) > 0$ if and only if $\ell \geq m$.*
2. *For all $n = 1, \ldots, L$, $\rho(F_n|\cup_\ell F_\ell) > 0$ iff $n \geq m$.*

An index m as in the above statement certainly exists, as $\bar{\ell} = L$ trivially belongs to the set on the right-hand side.

Proof: By Remark 2 part 4, $F_L \geq^\mu F_\ell$ for all $\ell = 1, \ldots, L$.

(1): from the definition of m, for $\ell \geq m$, $F_\ell \geq^\mu F_{\ell+1}$, and so $F_\ell \geq^\mu F_L$ by transitivity of $\geq^\mu$. Thus, for $\ell = m, \ldots, L$, $F_\ell =^\mu F_L$. By Proposition 1, $P_\mu(F_L)(F_\ell) > 0$ for such ℓ. It remains to be shown that these properties fail for $\ell < m$.

By contradiction, suppose $F_n \geq^\mu F_L$ for some $n < m$. The definition of μ-sequence and of the relation $\geq^\mu$ imply that $F_\ell \geq^\mu F_n$ for $\ell > n$, so by transitivity of $\geq^\mu$, $F_\ell \geq^\mu F_L$ for all $\ell \geq n$. In particular, $F_{m-1} \geq^\mu F_L$. Again, by the definition of μ-sequence, $F_L \geq^\mu F_{m-1}$; thus, $F_{m-1} =^\mu F_L$ and so by Proposition 1 $P_\mu(F_L)(F_{m-1}) > 0$. Since, by the definition of μ-sequence, $\mu(F_m|F_{m-1}) > 0$, the chain rule and the fact that ν extends μ imply that

$$0 < \mu(F_{m-1} \cap F_m | F_{m-1}) = \frac{P_\mu(F_L)(F_{m-1} \cap F_m)}{P_\mu(F_L)(F_{m-1})},$$

so $P_\mu(F_L)(F_{m-1} \cap F_m) > 0$. But since it was just shown that $P_\mu(F_L)(F_m) > 0$, the chain rule and the extension property also imply that

$$\mu(F_{m-1} \cap F_m | F_m) = \frac{P_\mu(F_L)(F_{m-1} \cap F_m)}{P_\mu(F_L)(F_m)} > 0,$$

which contradicts the definition of m.

Thus, $n < m$ implies not $F_n \geq^\mu F_L$, therefore not $F_n =^\mu F_L$; indeed, since $F_L \geq^\mu F_n$, $F_L >^\mu F_n$, and so $P_\mu(F_L)(F_n) = 0$ by Corollary 1 part 2.

(2): by Remark 2 part 5, there is a μ-sequence $F_{L+1}, \ldots, F_{L+M}$ such that $\{F_{L+1}, \ldots, F_{L+M}\}$ is the equivalence class of $\geq^\mu$ that contains F_L—and hence, by part (1) of this Lemma, $F_m, \ldots, F_{L-1}$ as well—and $F_{L+1} = F_L$. By Remark 2 part 2, $F_1, \ldots, F_{L+M}$ is a μ-sequence.

By construction, $\{F_m, \ldots, F_{L+M}\} = \{F_{L+1}, \ldots, F_{L+M}\}$. Since $\{F_{L+1}, \ldots, F_{L+M}\} = \{G \in \mathscr{F} : G \geq^\mu F_L, F_L \geq^\mu G\}$, by Lemma 2, since $F_{L+M} =^\mu F_L$, $\rho(\cdot | \cup_{\ell=m}^{L+M} F_\ell) = P_\mu(F_L)(\cdot)$. Hence, $\rho(F_n | \cup_{\ell=m}^{L+M} F_\ell) = P_\mu(F_L)(F_n) > 0$ for $n = m, \ldots, L$ from part 1 of this Lemma.

I claim that $\rho(\cup_{\ell=m}^{L+M} F_\ell | \cup_{\ell=1}^{L+M} F_\ell) > 0$. By contradiction, suppose this is not the case; let $n_0 \in \{2, \ldots, m\}$ be such that $\rho(\cup_{\ell=n_0}^{L+M} F_\ell | \cup_{\ell=1}^{L+M} F_\ell) = 0$ and $\rho(F_{n_0-1} | \cup_{\ell=1}^{L+M} F_\ell) > 0$. One such n_0 must exist, because by assumption $\rho(\cup_{\ell=m}^{L+M} F_\ell | \cup_{\ell=1}^{L+M} F_\ell) = 0$, and clearly $\rho(\cup_{\ell=1}^{L+M} F_\ell | \cup_{\ell=1}^{L+M} F_\ell) = 1$. By the chain rule, since $F_1, \ldots, F_L$ is a μ-sequence and ρ and μ agree on $\mathscr{F}$,

$$0 < \mu(F_{n_0} \cap F_{n_0-1} | F_{n_0-1}) = \frac{\rho(F_{n_0} \cap F_{n_0-1} | \cup_{\ell=1}^{L+M} F_\ell)}{\rho(F_{n_0-1} | \cup_{\ell=1}^{L+M} F_\ell)},$$

so $\rho(F_{n_0} | \cup_{\ell=1}^{L+M} F_\ell) \geq \rho(F_{n_0} \cap F_{n_0-1} | \cup_{\ell=1}^{L+M} F_\ell) > 0$: but this contradicts the definition of n_0, which proves the claim.

By Eq. (3), conclude that $\rho(F_n | \cup_{\ell=1}^{L+M} F_\ell) > 0$ for $n = m, \ldots, L$. Then also $\rho(\cup_{\ell=1}^{L} F_\ell | \cup_{\ell=1}^{L+M} F_\ell) > 0$, and so a further application of Eq. (3) yields $\rho(F_n | \cup_{\ell=1}^{L} F_\ell) > 0$ for $n = m, \ldots, L$.

Finally, suppose that $\rho(F_{n_1} | \cup_{\ell=1}^{L} F_\ell) > 0$ for some $n_1 < m$. I claim that then $\rho(F_n | \cup_{\ell=1}^{L} F_\ell) > 0$ for all $n = n_1, \ldots, m-1$. The claim is true by assumption fo $n = n_1$. Inductively, assume it is true for some $n-1 \geq n_1$. Since $F_1, \ldots, F_L$ is a μ-sequence, by Eq. (3)

$$0 < \mu(F_n \cap F_{n-1} | F_{n-1}) = \frac{\rho(F_n \cap F_{n-1} | \cup_{\ell=1}^{L} F_\ell)}{\rho(F_{n-1} | \cup_{\ell=1}^{L} F_\ell)},$$

so $\rho(F_n | \cup_{\ell=1}^{L} F_\ell) > 0$. Hence, in particular, $\rho(F_{m-1} | \cup_{\ell=1}^{L} F_\ell) > 0$. Again by Eq. (3) and the definition of μ-sequence,

$$0 < \mu(F_m \cap F_{m-1} | F_{m-1}) = \frac{\rho(F_m \cap F_{m-1} | \cup_{\ell=1}^{L} F_\ell)}{\rho(F_{m-1} | \cup_{\ell=1}^{L} F_\ell)},$$

so $\rho(F_m \cap F_{m-1} | \cup_{\ell=1}^{L} F_\ell) > 0$. Since, as was shown above, $\rho(F_m | \cup_{\ell=1}^{L}) > 0$, Eq. (3) implies that

$$\mu(F_{m-1} | F_m) = \mu(F_{m-1} \cap F_m | F_m) = \frac{\rho(F_m \cap F_{m-1} | \cup_{\ell=1}^{L} F_\ell)}{\rho(F_m | \cup_{\ell=1}^{L} F_\ell)} > 0.$$

which contradicts the definition of m. ■

Lemma 3 also yields the following property of full μ-sequences.

Lemma 4 *If a μ-sequence $F_1, \ldots, F_L \in \mathscr{F}$ is full, then for every $G \in \mathscr{F}$, $G =^\mu F_L$ implies $G \in \{F_1, \ldots, F_L\}$.*

Proof: Suppose $F_1, \ldots, F_L$ is a full sequence and consider $G \in \mathscr{F}$ with $G =^\mu F_L$. The *distance* from G to F_L is the length $M \geq 1$ of the shortest μ-sequence $G_1, \ldots, G_M$ with $G_1 = F_L$ and $G_M =$

G. If $M = 1$, then $G = G_M = G_1 = F_L$. Inductively, suppose $G' \in \{F_1, \dots, F_L\}$ for every $G' \in \mathscr{F}$ with $G' =^\mu F_L$ whose distance from F_L is at most M, and consider $G \in \mathscr{G}$ with $G =^\mu F_L$ and distance $M+1$ from F_L. Then there is a μ-sequence $G_1, \dots, G_{M+1}$ with $G_1 = F_L$ and $G_{M+1} = G$. By Lemma 3 part 1, since $G_1 =^\mu F_L =^\mu G =^\mu G_{M+1}$, $G_M =^\mu G_{M+1} =^\mu F_L$ as well; furthermore, $P_\mu(G_{M+1})(G_M) > 0$. Since $\mu(G_M \cap G_{M+1}|G_M) > 0$ by the definition of a μ-sequence, by Eq. (3) and the fact that ν extends μ it follows that $P_\mu(G_{M+1})(G_M \cap G_{M+1}) > 0$. Finally, since $P_\mu(G_{M+1})(G_{M+1}) > 0$ by Lemma 3 part 1, Eq. (3) and extension imply that $\mu(G_M \cap G_{M+1}|G_{M+1}) > 0$ as well. Hence, both $\mu(G_{M+1}|G_M) > 0$ and $\mu(G_M|G_{M+1}) > 0$.

Since G_M has distance at most M from F_L, by the inductive hypothesis there is $m \in \{1, \dots, L\}$ with $G_M = F_m$. Therefore, $\mu(G|F_m) = \mu(G_{M+1}|F_m) = \mu(G_{M+1}|G_M) > 0$. In addition, as was just shown, $\mu(F_m|G_{M+1}) = \mu(G_M|G_{M+1}) > 0$ as well. But then the definition of full sequence implies that $G = G_{M+1} \in \{F_1, \dots, F_L\}$. This completes the inductive step. ■

For every $s_{-i} \in S_{-i}$, let $[s_{-i}] = \{s_{-i}\} \times W$. Define the *$\mu$-support* of a μ-sequence $F_1, \dots, F_L$ as $\sigma_\mu(\cup_\ell F_\ell) = \cup\{[s_{-i}] : \rho([s_{-i}]| \cup_\ell F_\ell) > 0\}$; equivalently, $\sigma_\mu(\cup_\ell F_\ell) = \cup\{[s_{-i}] : P_\mu(F_L)([s_{-i}] \cap [\cup_\ell F_\ell]) > 0\}$.

Also observe that, by Remark 2 part 5, every equivalence class $\mathscr{C}$ for $\geq^\mu$ can be written as $\mathscr{C} = \cup_{\ell=1}^L F_\ell$ for some μ-sequence $F_1, \dots, F_L$. The definition of σ_μ only depends upon the union $\cup\mathscr{C} = \cup_\ell F_\ell$ (cf. Corollary 2), so one can write $\sigma_\mu(\cup\mathscr{C})$ without any ambiguity. Indeed in such case $\sigma_\mu(\cup\mathscr{C}) = \cup\{[s_{-i}] : P_\mu(F)([s_{-i}]) > 0\}$, where $F \in \mathscr{C}$ can be chosen arbitrarily.

I temporarily distinguish between the μ-support $\sigma_\mu(\cdot)$ and the support $\sigma(\cdot)$ introduced in Definition 6; however, the characterization of negligible events in both the necessity and sufficiency part of the proof immediately implies that $\sigma_\mu = \sigma$.

Lemma 5

1. *For all distinct equivalence classes $\mathscr{C}, \mathscr{D}$ of $\geq^\mu$, $\sigma_\mu(\cup\mathscr{C}) \cap \sigma_\mu(\cup\mathscr{D}) = \emptyset$.*

2. *Fix a μ-sequence $F_1, \dots, F_L$, and let m be as in Lemma 3. Then $\sigma_\mu(\cup_{\ell=1}^L F_\ell) = \sigma_\mu(\cup_{\ell=m}^L F_\ell)$,*

and $\sigma_\mu(\cup_\ell F_\ell) \subseteq \sigma_\mu(\cup \mathscr{C})$, where $\mathscr{C} = \{G \in \mathscr{F} : G =^\mu F_L\}$. Furthermore, for all other equivalence classes $\mathscr{D} \neq \mathscr{C}$ of $\geq^\mu$, $\sigma_\mu(\cup_\ell F_\ell) \cap \sigma_\mu(\cup \mathscr{D}) = \emptyset$, so $P_\mu(D)(\cup_\ell F_\ell) = 0$ for all $D \in \mathscr{D}$.

Proof: (1): fix $C \in \mathscr{C}$ and $D \in \mathscr{D}$ arbitrarily, so $\cup \mathscr{C} = B_\mu(C)$ and $\cup \mathscr{D} = B_\mu(D)$. Consider $s_{-i} \in S_{-i}$. If $[s_{-i}] \in \sigma_\mu(\cup \mathscr{C}) \cap \sigma_\mu(\cup \mathscr{D})$, then $P_\mu(C)([s_{-i}]) > 0$ and $P_\mu(D)([s_{-i}]) > 0$. Since $P_\mu(C)(\cup \mathscr{C}) = P_\mu(C)(B_\mu(C)) = 1$, it must be the case that $[s_{-i}] \cap (\cup \mathscr{C}) \neq \emptyset$. Since every $F \in \mathscr{C}$ is of the form $F = S_{-i}(I) \times W$ for some $I \in \mathscr{I}_i$, $[s_{-i}] \subseteq \cup \mathscr{C}$. Similarly, $[s_{-i}] \in \cup \mathscr{D}$. Let $F \in \mathscr{C}$, $G \in \mathscr{D}$ such that $[s_{-i}] \subseteq F$ and $[s_{-i}] \subseteq G$. Then $P_\mu(C)(G) \geq P_\mu(C)([s_{-i}]) > 0$ and $P_\mu(D)(F) \geq P_\mu(D)([s_{-i}]) > 0$. By Corollary 1 part (1), $G \geq^\mu C$ and $F \geq^\mu D$. But since C, F and, respectively, D, G are in the same equivalence class, also $C \geq^\mu F$ and $D \geq^\mu G$, so by transitivity $D \geq^\mu C$ and $C \geq^\mu D$, which contradicts the fact that $C \in \mathscr{C}$, $D \in \mathscr{D}$, and $\mathscr{C}$ and $\mathscr{D}$ are distinct equivalence classes.

(2): From Lemma 3 part 1, $\ell < m$ implies $P_\mu(F_L)(F_\ell) = 0$. Therefore, $P_\mu(F_L)([s_{-i}] \cap \cup_{\ell=1}^L F_\ell) \leq P_\mu(F_L)([s_{-i}] \cap \cup_{\ell=1}^{m-1} F_\ell) + P_\mu(F_L)([s_{-i}] \cap \cup_{\ell=m}^L F_\ell) = P_\mu(F_L)([s_{-i}] \cap \cup_{\ell=m}^L F_\ell)$. The reverse inequality holds by monotonicity of $P_\mu(F_L)(\cdot)$, so $P_\mu(F_L)([s_{-i}] \cap \cup_{\ell=1}^L F_\ell) = P_\mu(F_L)([s_{-i}] \cap \cup_{\ell=m}^L F_\ell)$. By definition, this implies that $\sigma_\mu(\cup_{\ell=1}^L F_\ell) = \sigma_\mu(\cup_{\ell=m}^L F_\ell)$.

If $\mathscr{C}$ is the equivalence class for $\geq^\mu$ that contains F_L, $\sigma_\mu(\cup \mathscr{C}) = \cup\{[s_{-i}] : P_\mu(F_L)([s_{-i}]) > 0\}$. Hence, $\sigma_\mu(\cup_{\ell=1}^L F_\ell) = \sigma_\mu(\cup_{\ell=m}^L F_\ell) = \cup\{[s_{-i}] : P_\mu(F_L)([s_{-i}] \cap \cup_{\ell=m}^L F_\ell) > 0\} \subseteq \cup\{[s_{-i}] : P_\mu(F_L)([s_{-i}]) > 0\} = \sigma_\mu(\cup \mathscr{C})$.

Now let $\mathscr{D}$ be another equivalence class of $\geq^\mu$. By part 1 of this Lemma, $\sigma_\mu(\cup \mathscr{C}) \cap \sigma_\mu(\cup \mathscr{D}) = \emptyset$. It follows that $\sigma_\mu(\cup_\ell F_\ell) \cap \sigma_\mu(\cup \mathscr{D}) \subseteq \sigma_\mu(\cup \mathscr{C}) \cap \sigma_\mu(\cup \mathscr{D}) = \emptyset$ for any $\mathscr{D} \neq \mathscr{C}$. The last claim follows from the observation that $P_\mu(D)(\sigma_\mu(\cup \mathscr{D})) = \sum_{s_{-i}: P_\mu(D)([s_{-i}])>0} P_\mu(D)([s_{-i}]) = \sum_{s_{-i} \in S_{-i}} P_\mu(D)([s_{-i}]) - \sum_{s_{-i}: P_\mu(D)([s_{-i}])=0} P_\mu(D)([s_{-i}]) = P_\mu(D)(\Omega) - 0 = 1$ for any $D \in \mathscr{D}$. ■

B Characterization: necessity

Assume throughout that μ is CCPS and $\succ$ is as in Def. 4. By Lemma 2, ρ extends μ and $P_\mu(\cdot 0)$ to $\mathscr{F}_\mu$. Recall that, in particular, for all $F \in \mathscr{F}$, $P_\mu(F)(\cdot) = \rho(\cdot|B_\mu(F))$.

This lemma characterizes negligibility for a conditioning event in terms of the CPS ν.

Lemma 6 *Consider $N \in \Sigma$ and a μ-sequence $F_1, \dots, F_L$. Then $N \in \Sigma$ is negligible given $\cup_\ell F_\ell$ iff $P_\mu(F_L)(N \cap (\cup_\ell F_\ell)) = 0$. In particular, N is negligible given $F \in \mathscr{F}$ iff $\mu(N|F) = 0$.*

Proof: Suppose $P_\mu(F_L)(N \cap (\cup_\ell F_\ell)) = 0$. If $f, g \in \mathscr{A}$ satisfy $f(\omega) \succ g(\omega)$ for $\omega \notin N$, in particular this holds for $\omega \in (\cup_\ell F_\ell) \setminus N$. Since $P_\mu(F_L)((\cup_\ell F_\ell) \setminus N) = P_\mu(F_L)(\cup_\ell F_\ell) - P_\mu(F_L)(N \cap (\cup_\ell F_\ell)) = P_\mu(F_L)(\cup_\ell F_\ell)$, and $P_\mu(F_L)(\cup_\ell F_\ell) \geq P_\mu(F_L)(F_L) > 0$ by Lemma 1, $\int u \circ f(\cup_\ell F_\ell) g \, dP_\mu(F_L)(\cdot) > \int u \circ g \, dP_\mu(F_L)(\cdot)$.

Now suppose $u(g(\omega)) < u(f(\omega))$ for some $\omega \in \cup_\ell F_\ell$ [recall that the objective is to rank $f(\cup_\ell F_\ell) g$ and g]. By assumption and the fact that u represents $\succ$ on X, it must be that $\omega \in N \cap (\cup_\ell F_\ell)$. Hence $\omega \in F_\ell$ for some ℓ, $F_L \geq^\mu F_\ell$, and as was just argued, $\int u \circ f(\cup_\ell F_\ell) g \, dP_\mu(F_L)(\cdot) > \int u \circ g \, dP_\mu(F_L)(\cdot)$. Thus, $f(\cup_\ell F_\ell) g \succ g$, i.e., $f \succ_{\cup_\ell F_\ell} g$. Since f and g were arbitrary, This implies that N is negligible given $\cup_\ell F_\ell$.

For the converse, it is enough to consider the case $N \subseteq \cup_\ell F_\ell$: for general $N \in \Sigma$, if $N_1 = N \cap (\cup_\ell F_\ell)$ and $N_2 = N \setminus (\cup_\ell F_\ell)$, then $P_\mu(F_L)(N \cap (\cup_\ell F_\ell)) = P_\mu(F_L)(N_1 \cap (\cup_\ell F_\ell)) + P_\mu(F_L)(N_2 \cap (\cup_\ell F_\ell)) = P_\mu(F_L)(N_1 \cap (\cup_\ell F_\ell))$, and $N_1 \subseteq \cup_\ell F_\ell$.

Suppose that $P_\mu(F_L)(N) > 0$. As argued above, $P_\mu(F_L)(\cup_\ell F_\ell) > 0$. Choose x, y such that $x \succ y$, and

$$\alpha \in \left(0, \frac{P_\mu(F_L)(N)}{P_\mu(F_L)(\cup_\ell F_\ell)}\right).$$

Let $f = \alpha x + (1-\alpha) y$ and $g = x N y$; thus, $f(\omega) = \alpha x + (1-\alpha) y \prec x = g(\omega)$ for $\omega \in N$, and

$f(\omega) = \alpha x + (1-\alpha)y \succ y = g(\omega)$ for $\omega \notin N$. Moreover,

$$\int u \circ f(\cup_\ell F_\ell) g \, d P_\mu(F_L)(\cdot) = P_\mu(F_L)(\cup_\ell F_\ell)[\alpha u(x) + (1-\alpha)u(y)] + [1 - P_\mu(F_L)(\cup_\ell F_\ell)]u(y) <$$

$$< P_\mu(F_L)(\cup_\ell F_\ell)\left[\frac{P_\mu(F_L)(N)}{P_\mu(F_L)(\cup_\ell F_\ell)} u(x) + \left(1 - \frac{P_\mu(F_L)(N)}{P_\mu(F_L)(\cup_\ell F_\ell)}\right) u(y)\right] + (1 - P_\mu(F_L)(\cup_\ell F_\ell))u(y) =$$

$$= P_\mu(F_L)(N)u(x) + [1 - P_\mu(F_L)(N)]u(y) = \int u \circ g \, d P_\mu(F_L)(\cdot);$$

the inequality follows from the choice of α, and is strict because $P_\mu(F_L)(\cup_\ell F_\ell) > 0$.

By contradiction, assume N is negligible given $\cup_\ell F_\ell$. Then in particular $f \succ_{\cup_\ell F_\ell} g$, i.e., $f(\cup_\ell F_\ell) g \succ g$. By definition, for every $\omega \in N \subseteq \cup_\ell F_\ell$, there are $F^\omega, G^\omega \in \mathscr{F}$ satisfy $\omega \in G^\omega$, $F^\omega \geq^\mu G^\omega$, and $\int u \circ f(\cup_\ell F_\ell) g \, d\nu \cdot |B_\mu(F^\omega)) > \int u \circ g \, d P_\mu(F^\omega)(\cdot)$. Then $P_\mu(F^\omega)((\cup_\ell F_\ell) \setminus N) > 0$, so by Lemma 1 $F_L \geq^\mu F^\omega$. Since $\int u \circ f(\cup_\ell F_\ell) g \, d\nu \cdot |B_\mu(F_L)) < \int u \circ g \, d P_\mu(F_L)(\cdot)$, $F_L >^\mu F^\omega$.

Now $N = \cup_{\omega \in N}(N \cap G^\omega)$, which is a union of finitely many elements because $\mathscr{F}$ is finite. Thus, $0 < P_\mu(F_L)(N) \leq \sum_{\omega \in N} P_\mu(F_L)(N \cap G^\omega)$, which implies that $P_\mu(F_L)(G^{\bar\omega}) > 0$ for some $\bar\omega \in N$. But then $F_L >^\mu F^{\bar\omega} \geq^\mu G^{\bar\omega} \geq^\mu F_L$, contradiction. Therefore, not $f(\cup_\ell F_\ell) g \succ g$, i.e., not $f \succ_{\cup_\ell F_\ell} g$, so N is not negligible given $\cup_\ell F_\ell$.

The last claim follows by noting that any $F \in \mathscr{F}$ is a degenerate μ-sequence of length $L = 1$, so N is negligible given F iff $P_\mu(F)(N \cap F) = 0$. By Lemma 1, $P_\mu(F)(F) > 0$, so by Eq. (3) $\mu(N \cap F | F) = P_\mu(F)(N \cap F)/P_\mu(F)(F)$. Hence, the claim holds for all $N \subseteq F$. For general N, the claim holds because $\mu(N|F) = \mu(N \cap F|F)$. ■

It follows that a n-sequence in the sense of Definition 6 is a μ-sequence, and conversely. Similarly, a full seqeunce is a full μ-sequence, and conversely. Finally, for any μ-sequence $F_1, \ldots, F_L$, $\sigma(\cup_\ell F_\ell) = \cup\{[s_{-i}] : P_\mu(F_L)([s_{-i}] \cap (\cup_\ell F_\ell)) > 0\} = \sigma_\mu(\cup_\ell F_\ell)$, where σ_μ is defined in Appendix A.

Throughout the remainder of this Section, I will not distinguish between n-sequences and μ-sequences, or between σ and σ_μ.

In particular, Lemma 5 implies

Lemma 7 *For every $F \in \mathscr{F}$, $\succ_{\sigma(F)}$ is an EU preference relation, represented by u and $\mu(\cdot|F)$.*

Proof: Consider two acts $f, g \in \mathscr{A}$. By Lemma 5 part 2, applied to the degenerate μ-sequence $F_1 = F$, if $G \in \mathscr{F}$ is such that not $F =^{\mu} G$, then $\int u \circ f\sigma(F)g\, dP_{\mu}(G)(\cdot) = \int u \circ g\, dP_{\mu}(G)(\cdot)$, because $P_{\mu}(G)(\sigma(F)) = 0$ and $f\sigma(F)g$ agrees with g outside the event $\sigma(F)$.

Suppose first that $\int u \circ f\sigma(F)g\, dP_{\mu}(F_L)(\cdot) > \int u \circ g\, dP_{\mu}(F_L)(\cdot)$. Suppose further that $u(f(\omega)) < u(g(\omega))$ for some $\omega \in \sigma(F)$. Then $\omega \in F$, and trivially $F \geq^{\mu} F$. Therefore $f\sigma(F)g \succ g$, i.e., $f \succ_{\sigma(F)} g$.

Conversely, suppose that $f \succ_{\sigma(F)} g$, i.e., $f\sigma(F)g \succ g$. Then there must be G with $\int u \circ f\sigma(F)g\, dP_{\mu}(G)(\cdot) \neq \int u \circ g\, dP_{\mu}(G)(\cdot)$. By the argument given above, $F =^{\mu} G$ for any such G, so one can take $G = F$. Next, by contradiction, suppose $\int u \circ f\sigma(F)g\, dP_{\mu}(F_L)(\cdot) < \int u \circ g\, dP_{\mu}(F_L)(\cdot)$. Then there must be $\omega \in \sigma(F)$ with $u(f(\omega)) < u(g(\omega))$. Since $f\sigma(F)g \succ g$, there must be $F', G' \in \mathscr{F}$ with $\omega \in G'$, $F' \geq^{\mu} G'$, and $\int u \circ f\sigma(F')g\, dP_{\mu}(F')(\cdot) > \int u \circ g\, dP_{\mu}(F)(\cdot)$. But, as was just argued, $F =^{\mu} F'$ for any such F', and so $\int u \circ f\sigma(F)g\, dP_{\mu}(F')(\cdot) = \int u \circ f\sigma(F)g\, dP_{\mu}(F)(\cdot) < \int u \circ g\, dP_{\mu}(F)(\cdot) = \int u \circ g\, dP_{\mu}(F')(\cdot)$, contradiction. Therefore, $\int u \circ f\sigma(F)g\, dP_{\mu}(F_L)(\cdot) > \int u \circ g\, dP_{\mu}(F_L)(\cdot)$. ■

It is now possible to **verify that Axioms 1–9 hold**. The following simple alternative characterization of structural preferences is convenient:

Observation 1 $f \succ^{u,\mu} g$ iff (1) there is $F^* \in \mathscr{F}$ with $\int u \circ f\, dP_{\mu}(F^*) \neq \int u \circ g\, dP_{\mu}(F^*)$, and (2) for every $\omega \in \Omega$ with $u(f(\omega)) < u(g(\omega))$, there are $F, G \in \mathscr{F}$ such that $\omega \in G$, $F \geq^{\mu} G$, and $\int u \circ f\, dP_{\mu}(F) \neq \int u \circ g\, dP_{\mu}(F)$.

The proof is immediate. I employ this result throughout the remainder of this section.

Assume that $\succ$ is defined via Definition 5. The irreflexivity part of Axiom 1, as well as Axioms 2 (Prize Negative Transitivity) and 3 (Independence) are straightforward to verify.

For Transitivity, suppose that $f \succ g$ and $g \succ h$. By contradiction, suppose that $\mathrm{E}_{\nu(\cdot|F)} u \circ$

$f = \mathrm{E}_{\mu(\cdot|F)} u \circ h$ for all $F \in \mathscr{F}$. There must be $G_1 \in \mathscr{F}$ such that $\mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ f \neq \mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ g$, so that also $\mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ g \neq \mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ h$. Suppose $\mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ f < \mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ g$: then there is ω such that $u(f(\omega)) < u(g(\omega))$ and $P_\mu(G_1)(\{\omega\}) > 0$. Since $f \succ g$, there are F, G such that $F \geq^\mu G$ and $\omega \in G$, such that $\mathrm{E}_{P_\mu(F)(\cdot)} u \circ f > \mathrm{E}_{P_\mu(F)(\cdot)} u \circ g$. Then $P_\mu(G_1)(G) > 0$, so by Lemma 1 $F \geq^\mu G \geq^\mu G_1$. Since $\mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ f < \mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ g$, $F >^\mu G_1$. Let $G_2 = F$. Then, by assumption, also $\mathrm{E}_{P_\mu(G_2)(\cdot)} u \circ h > \mathrm{E}_{P_\mu(G_2)(\cdot)} u \circ g$, so there is ω such that $u(h(\omega)) > u(g(\omega))$ and $P_\mu(G_2)(\{\omega\}) > 0$. Since $g \succ h$, a similar argument yields $G_3 >^\mu G_2$ such that $\mathrm{E}_{P_\mu(G_3)(\cdot)} u \circ f < \mathrm{E}_{P_\mu(G_3)(\cdot)} u \circ g$, etc. This process cannot continue indefiinitely because $\mathscr{F}$ is finite. Therefore, it cannot be the case that $\mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ f < \mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ g$. If instead $\mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ f > \mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ g$, then also $\mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ h > \mathrm{E}_{P_\mu(G_1)(\cdot)} u \circ g$, and a similar argument again leads to a contradiction. Therefore, there must be F with $\mathrm{E}_{P_\mu(F)(\cdot)} u \circ f \neq \mathrm{E}_{\nu(\cdot|B_\mu(F))} u \circ h$.

Now suppose that $u(f(\omega)) < u(h(\omega))$ for some ω. There are two cases. First, suppose $u(g(\omega)) \leq u(f(\omega))$. Then $u(g(\omega)) < u(h(\omega))$, so $g \succ h$ implies that there are F_1, G_1 with $\omega \in G_1$, $F_1 \geq^\mu G_1$, and $\mathrm{E}_{P_\mu(F_1)(\cdot)} u \circ g > \mathrm{E}_{P_\mu(F_1)(\cdot)} u \circ h$. If also $\mathrm{E}_{P_\mu(F_1)(\cdot)} u \circ f \geq \mathrm{E}_{P_\mu(F_1)(\cdot)} u \circ g$, the proof is complete. Otherwise, $\mathrm{E}_{P_\mu(F_1)(\cdot)} u \circ f < \mathrm{E}_{P_\mu(F_1)(\cdot)} u \circ g$, so there must be ω^1 such that $u(f(\omega^1)) < u(g(\omega^1))$ and $P_\mu(F_1)(\{\omega^1\}) > 0$. Then $f \succ g$ implies that there are F_2, G_2 with $\omega^1 \in G^2$, $F^2 \geq^\mu G_2$, and $\mathrm{E}_{P_\mu(F_2)(\cdot)} u \circ f > \mathrm{E}_{P_\mu(F_2)(\cdot)} u \circ g$. Since $P_\mu(F_1)(G^2) \geq P_\mu(F_1)(\{\omega^1\}) > 0$, $F^2 \geq^\mu G^2 \geq^\mu F^1$. Moreover, if also $\mathrm{E}_{P_\mu(F_2)(\cdot)} u \circ g \geq \mathrm{E}_{P_\mu(F_2)(\cdot)} u \circ h$, the proof is complete. Otherwise, $\mathrm{E}_{P_\mu(F_2)(\cdot)} u \circ g < \mathrm{E}_{P_\mu(F_2)(\cdot)} u \circ h$, so $F_1 \neq F^2$ and therefore $F_2 >^\mu F_1$, and $u(g(\omega^2)) < u(h(\omega^2))$ for some $\omega^2 \in F^2$. Again, $g \succ h$ implies that there are F_3, G_3 with $\omega^2 \in G_3$, $F_3 \geq^\mu G_3$, and $\mathrm{E}_{P_\mu(F_3)(\cdot)} u \circ g > \mathrm{E}_{P_\mu(F_3)(\cdot)} u \circ h$. Either also $\mathrm{E}_{P_\mu(F_3)(\cdot)} u \circ f \geq \mathrm{E}_{P_\mu(F_3)(\cdot)} u \circ g$, and the proof is complete, or the argument for F_1 can be repeated verbatim. Since $\mathscr{F}$ is finite, this inductive construction must stop at some round k, and the set F_k satisfies $F_k \geq^\mu G_1$ and $\mathrm{E}_{P_\mu(F_k)(\cdot)} u \circ f > \mathrm{E}_{P_\mu(F_k)(\cdot)} u \circ h$. In the second case, $u(g(\omega)) > u(f(\omega))$. Then $f \succ g$ yields F_1, G_1 with $\omega \in G_1$, $F_1 \geq^\mu G_1$, and $\mathrm{E}_{P_\mu(F_1)(\cdot)} u \circ f > \mathrm{E}_{P_\mu(F_1)(\cdot)} u \circ g$. An analogous argument yields an event F_k with the required properties.

For Axioms 5 and 6, fix $F \in \mathscr{F}$. Then, by Proposition 1, $P_\mu(F)(F) > 0$. Now consider $x, y \in X$. If $u(x) > u(y)$, then $\mathrm{E}_{P_\mu(G)(\cdot)} u \circ xFy \geq u(y) = \mathrm{E}_{P_\mu(G)(\cdot)} u(y)$ for all $G \in \mathscr{F}$; furthermore, $\mathrm{E}_{P_\mu(F)(\cdot)} u \circ$

$xFy > \mathrm{E}_{P_\mu(F)(\cdot)} u(y)$. Hence $xFy \succ x$, so $x \succ_F y$. Conversely, suppose $x \succ_F y$, i.e., $xFy \succ y$. By definition, $\mathrm{E}_{P_\mu(G)(\cdot)} u \circ xFy \geq u(y) \neq \mathrm{E}_{P_\mu(G)(\cdot)} u(y)$ for some $G \in \mathscr{F}$. This implies $u(x) \neq u(y)$. By contradiction, suppose $u(x) < u(y)$. Then $\mathrm{E}_{P_\mu(G)(\cdot)} u \circ xFy \geq u(y) \leq \mathrm{E}_{P_\mu(G)(\cdot)} u(y)$ for every $G \in \mathscr{F}$, and for $\omega \in F$, $u \circ xFy(\omega) = u(x) < u(y) = u \circ y(\omega)$; thus, not $xFy \succ y$. Therefore, $u(x) > u(y)$. Therefore, $x \succ_F y$ iff $u(x) > u(y)$. Since u is a non-constant, affine utility function, Axioms 5 and 6 hold.

For Axioms 7 and 8, Lemma 7 shows that, for every $F \in \mathscr{F}$, $\succ_{\sigma(F)}$ is an EU preference relation; hence, it is negatively transitive and Archimedean, so the Axioms hold.

Finally, for Axiom 4 (Monotonicity), consider x, y, E, f, g as in the statement. Note that $u(x) \geq u(y)$ because u represents $\succ$ on X. Suppose that $yEf \succ g$ and consider ω such that $u \circ xEf(\omega) < u(g(\omega))$. If $\omega \notin E$, then $xEf(\omega) = f(\omega)$, and so also $u \circ yEf(\omega) < u(g(\omega))$. If instead $\omega \in E$, then $u(x) < u(g(\omega))$ and so a fortiori $u(y) < u(g(\omega))$; thus, again, $u \circ yEf(\omega) < u(g(\omega))$. In either case, $yEf \succ g$ implies that there are $F, G \in \mathscr{F}$ with $F \geq^\mu G$, $\omega \in G$, and $\mathrm{E}_{P_\mu((F)(\cdot)} u \circ yEf > \mathrm{E}_{P_\mu(F)(\cdot)} u \circ g$; since $u(x) \geq u(y)$, $\mathrm{E}_{P_\mu((F)(\cdot)} u \circ xEf > \mathrm{E}_{P_\mu(F)(\cdot)} u \circ g$. Finally, $yEf \succ h$ implies that $\mathrm{E}_{P_\mu((F)(\cdot)} u \circ yEf \neq \mathrm{E}_{P_\mu(F)(\cdot)} u \circ g$ for some $F \in \mathscr{F}$. If $\mathrm{E}_{P_\mu((F)(\cdot)} u \circ yEf > \mathrm{E}_{P_\mu(F)(\cdot)} u \circ g$, then also $\mathrm{E}_{P_\mu((F)(\cdot)} u \circ xEf > \mathrm{E}_{P_\mu(F)(\cdot)} u \circ g$. Otherwise, there must be ω with $u \circ yEf(\omega) < u(g(\omega))$, and the argument just given implies that there is $F' \in \mathscr{F}$ with $\mathrm{E}_{P_\mu((F')(\cdot)} u \circ xEf > \mathrm{E}_{P_\mu(F')(\cdot)} u \circ g$. Thus, $xEf \succ g$. The argument for $g \succ xEf \Rightarrow g \succ yEf$ is analogous.

Finally, for Axiom 9, suppose that $f \succ g$ and $f(\omega) \prec g\omega)$ for all $\omega \in E \in \Sigma$. If $E = \emptyset$, by Definition 4 $f \succ g$ implies that $\mathrm{E}_{P_\mu(F)(\cdot)}[u \circ f - u \circ g] > 0$ for some $F \in \mathscr{F}$. Let $x, y \in X$ be such that $\mathrm{E}_{P_\mu(F)(\cdot)} u \circ f > u(x) > u(y) > \mathrm{E}_{P_\mu(F)(\cdot)} u \circ g$, and let $F_1, \ldots, F_L$ be the μ-sequence consisting of elements of the $\geq^\mu$-equivalence class of F, whose existence is guaranteed by Remark 2 part 5. If $\mu(G|F_\ell) > 0$ and $\mu(G|F_\ell) > 0$ for some $G \in \mathscr{F}$, then $G =^\mu F_\ell =^\mu F_L$, so $G \in \{F_1, \ldots, F_L\}$ by construction. Thus, $F_1, \ldots, F_L$ is a full μ-sequence, hence a full sequence. Since $F = F_L$, $\mathrm{E}_{P_\mu(F)(\cdot)} u \circ [f B_\mu(F) g] > \mathrm{E}_{P_\mu(F)(\cdot)} u \circ [x B_\mu(F) g] > \mathrm{E}_{P_\mu(F)(\cdot)} u \circ [x B_\mu(F) g] > \mathrm{E}_{P_\mu(F)(\cdot)} u \circ g$. Furthermore, if $u \circ [f B_\mu(F) g](\omega) < u \circ [x B_\mu(F) g]$ or $u \circ [y B_\mu(F) g] < u \circ g$, then $\omega \in B_\mu(F)$, so $\omega \in F_\ell$ for some $\ell \in \{1, \ldots, F_L\}$. Hence, $F_L \geq^\mu F_\ell$. Thus, Definition 4 implies $f B_\mu(F) g \succ x B_\mu(F) g \succ y B_\mu(F) g \succ g$, i.e., $f \succ_{\cup_\ell F_\ell} x \succ_{\cup_\ell F_\ell} y \succ_{\cup_\ell F_\ell} g$, as required.

If instead $E \neq \emptyset$, fix $\omega \in E$. by Definition 4 $f \succ g$ implies that there are $F, G \in \mathscr{F}$ with $F \geq^\mu G$, $\omega \in G$, and $\mathrm{E}_{P_\mu(F)(\cdot)}[u \circ f - u \circ g] > 0$. Let $F_1, \ldots, F_L$ be a μ-sequence such that $F_1 = G$ and $F_L = F$. For every $\ell = 1, \ldots, L$, let $F_1^\ell, \ldots, F_{L(\ell)}^\ell$ be a μ-sequence with $F_1^\ell = F_{L(\ell)}^\ell = F_\ell$, per Remark 2 part 5, Then, similarly to the case $E = \emptyset$,

$$F_1^1, \ldots, F_{L(1)}^1, F_1^2, \ldots, F_{L(L)}^L$$

is a full sequence, and $F_{L(L)}^L = F_L =^\mu F$. A slight modification of the preceding argument then yields $f \succ_{\cup_{\ell=1}^L \cup_{k=1}^{L(\ell)} F_k^\ell} x \succ_{\cup_{\ell=1}^L \cup_{k=1}^{L(\ell)} F_k^\ell} y \succ_{\cup_{\ell=1}^L \cup_{k=1}^{L(\ell)} F_k^\ell} g$, as required. This completes the proof.

C Sufficiency: Preliminaries

Begin by proving Remark 1; the proof is essentially standard, but it is provided here to emphasize that it only relies on the Independence axiom.

Proof: Fix $f, g, k, k', E \in \mathscr{A}$ as in the Remark. By Independence (Axiom 3),

$$fEk \succ gEk \quad \Leftrightarrow \quad \frac{1}{2} fEk + \frac{1}{2} k' \succ \frac{1}{2} gEk + \frac{1}{2} k',$$

and similarly

$$fEk' \succ gEk' \quad \Leftrightarrow \quad \frac{1}{2} fEk' + \frac{1}{2} k \succ \frac{1}{2} gEk' + \frac{1}{2} k.$$

Now observe that $\frac{1}{2} fEk + \frac{1}{2} k' = \frac{1}{2} fEk' + \frac{1}{2} k$: in every state $\omega \in E$

$$\frac{1}{2} fEk(\omega) + \frac{1}{2} k'(\omega) = \frac{1}{2} f(\omega) + \frac{1}{2} x = \frac{1}{2} fEk'(\omega) + \frac{1}{2} k(\omega),$$

and in every state $\omega \notin E$

$$\frac{1}{2} fEk(\omega) + \frac{1}{2} k'(\omega) = \frac{1}{2} k(\omega) + \frac{1}{2} k'(\omega) = \frac{1}{2} k(\omega) + \frac{1}{2} k'(\omega) = \frac{1}{2} k(\omega) + \frac{1}{2} k'(\omega) = \frac{1}{2} k(\omega) + \frac{1}{2} fEk'(\omega).$$

Similarly $\frac{1}{2} gEk + \frac{1}{2} k' = \frac{1}{2} gEk' + \frac{1}{2} k$. The claim follows. ■

Next, I relate prior and conditional preferences over X. For $E \in \Sigma$, say that $\succ_E$ is **non-degenerate** if there exist $x', y' \in X$ with $x' \succ_E y'$.

Lemma 8 *Assume Axioms 1–4 and 6. Consider $E \in \Sigma$ such that $\succ_E$ is non-degenerate. Then, for every $x, y \in X$, $x \succ_E y$ if and only if $x \succ y$.*

Proof: Fix $f \in \mathscr{A}$. Then $x \succ_E y$ iff $xEf \succ yEf$ for any $x, y \in X$, including x', y'.

If $x \succ_E y$ and not $x \succ y$, then $xEf \succ yEf$ and not $x \succ y$, so Axiom 4 (Monotonicity) implies $xEf \succ xEf$, which contradicts Irreflexivity in Axiom 1. Thus, $x \succ y$.

Coonversely, suppose $x \succ y$. By assumption $x'Ef \succ y'Ef$.

Case 1: not $x' \succ x$. Then Axiom 4 (Monotonicity) yields $xEf \succ y'Ef$. If also not $y \succ y'$, then the same Axiom yields $xEf \succ yEf$. Otherwise, $y \succ y'$. By Axiom 6 (Prize Continuity), there is $\alpha \in (0,1)$ such that $\alpha x + (1-\alpha)y' \succ y$. Then by Axiom 3 (Independence) $xEf \succ [\alpha x + (1-\alpha)y')]Ef$, and since Irreflexivity in Axiom 1 implies not $y \succ \alpha x + (1-\alpha)y'$, Axiom 4 implies $xEf \succ yEf$.

Case 2a: $x' \succ x$ and not $y \succ y'$. Then Axiom 4 (Monotonicity) yields $x'Ef \succ yEf$. Furthermore, by Axiom 6 (Prize Continuity), $x' \succ x$ and $x \succ y$ imply that there is $\alpha \in (0,1)$ such that $x \succ \alpha x' + (1-\alpha)y$. From Axiom 6 again, $x'Ef \succ yEf$ implies $[\alpha x' + (1-\alpha)y]Ef \succ yEf$. Irreflexivity implies not $\alpha x' + (1-\alpha)y \succ x$, so Axiom 4 finally implies $xEf \succ yEf$.

Case 2b: $x' \succ x$ and $y \succ y'$. By standard arguments, there is α such that not $\alpha x' + (1-\alpha)y' \succ y$ and not $y \succ \alpha x' + (1-\alpha)y'$.[4] By Axiom 3, $x'Ef \succ [\alpha x' + (1-\alpha)y']Ef$; since not $y \succ \alpha x' + (1-$

[4]For every $\lambda, \gamma \in [0,1]$, Independence and $x' \succ y'$ imply that $\lambda x' + (1-\lambda)[\gamma x' + (1-\gamma)y'] \succ \lambda y' + (1-\lambda)[\gamma x' + (1-\gamma)y']$. If $1 \geq \alpha > \beta \geq 0$, then letting $\lambda = \alpha - \beta$ and $\gamma = \frac{\beta}{1-\alpha+\beta}$ yields $\alpha x' + (1-\alpha)y' \succ \beta x' + (1-\beta)y'$ (if $\alpha = 1$ and $\beta = 0$ then γ is not well-defined, but the conclusion still holds trivially). Now let $U = \{\beta : \beta x' + (1-\beta)y' \succ y\}$ and $L = \{y \succ \beta x' + (1-\beta)y'\}$. Clearly $1 \in U$ and $0 \in L$. If $\beta \in U$ and $1 \geq \beta' > \beta$, then $\beta' \in U$; similarly, $\alpha \in L$ and $\alpha' \in [0, \alpha)$ imply $\alpha' \in L$. By Irreflexivity, $U \cap L = \emptyset$. Let $\alpha = \inf U$. If $\alpha \in U$, then $\alpha x' + (1-\alpha)y' \succ y \succ y'$ and Axiom 6 (Prize Continuity) imply that there is $\lambda \in (0,1)$ with $\lambda[\alpha x' + (1-\alpha)y'] + (1-\lambda)y' \succ y$, so $\lambda\alpha \in U$, which contradicts the definition of α. If $\alpha \in L$, then $x' \succ y \succ \alpha x' + (1-\alpha)y'$ and Axiom 6 imply that there is $\lambda \in (0,1)$ with $y \succ \lambda x' + (1-\lambda)[\alpha x' + (1-\alpha)y']$, so $\lambda + (1-\lambda)\alpha \in L$; but $\lambda + (1-\lambda)\alpha > \alpha$, so there is $\gamma \in (\alpha, \lambda + (1-\lambda)\alpha)$ with

$\alpha)y'$, Axiom 4 yields $x'Ef \succ yEf$. Again, by standard arguments there is $\beta \in (0,1)$ such that not $\beta x' + (1-\beta)y \succ x$ and not $x \succ \beta x' + (1-\beta)y$. By Axiom 3, $[\beta x' + (1-\beta)y]Ef \succ yEf$, and since not $\beta x' + (1-\beta)y) \succ x$, Axiom 4 finally yields $xEf \succ yEf$. ■

Remark 3 (Generalized Monotonicity) *Assume Axiom 4. For all $f,g,h \in \mathscr{A}$, if, for all ω, not $g(\omega) \succ f(\omega)$ [resp. not $f(\omega) \succ g(\omega)$], then $g \succ h$ implies $f \succ h$ [resp. $h \succ g$ implies $h \succ f$]*

Proof: Since f and g take on finitely many distinct values, there is a finite partition $E_1, \dots, E_N$ of Ω such that, for all $n = 1, \dots, N$ and $\omega, \omega' \in E_n$, $f(\omega) = f(\omega') \equiv x_n$ and $g(\omega) = g(\omega') \equiv y_n$. Assume that, for every n not $y_n \succ x_n$, and $g \succ h$. I show that then $f \succ h$; the other statement is proved similarly. Let $f^0 = g$. Inductively, for $n = 1, \dots, N$, let $f^n = x_n E_n f^{n-1}$. For $n = 0$, $f^0 = g$, so trivially $g \succ h$ implies $f \succ h$. Inductively, suppose that $f^{n-1} \succ h$ for some $n \in \{1, \dots, N-1\}$. Since $f^n = x_n E_n f^{n-1}$ and $f^{n-1} = y_n E_n f^{n-1}$, not $y_n \succ x_n$ and $f^{n-1} \succ h$ imply $f^n \succ h$ by Axiom 4 (Monotonicity). Since $f^N = f$, $f \succ h$. ■

Observation 2 (State Independence) Assume Axioms 1–4 and 6. If E is not Savage-null for $\succ$, then for all $x, y \in X$ and $f \in \mathscr{A}$, $x \succ y$ if and only if $xEf \succ yEf$.

Proof: If E is not Savage-null for $\succ$, there must be $f, g \in \mathscr{A}$ such that $f(\omega) = g(\omega)$ for $\omega \notin E$ and $f \succ g$. Let $x', y' \in X$ be such that not $f(\omega) \succ x'$ and not $y' \succ g(\omega)$ for all $\omega \in E$. Then Remark 3, invoked twice, implies that $x'Ef \succ y'Eg = y'Ef$. By Definition 5, $x' \succ_E y'$. Hence Lemma 8 implies that $x \succ y$ iff $x \succ_E y$; by Definition 5, this is equivalent to the claim. ■

Observation 3 (Null Complement) Fix an event $E \in \Sigma$ and acts $f, g \in \mathscr{A}$. If $f(\omega) = g(\omega)$ for all $\omega \in E$, then for every $h \in \mathscr{A}$, $f \succ_E h$ (resp. $h \succ_E k$) iff $g \succ_E h$ (resp. $h \succ_E g$).

$\gamma \in U$. But $\lambda + (1-\lambda)\alpha \in L$ and $\lambda + (1-\lambda)\alpha > \gamma$ imply $\gamma \in L$, contradiction. Thus, $\alpha \notin U \cup L$, as required.

(This implies that $\Omega\setminus E$ is Savage-null for $\succ_E$, but is a strictly stronger statement. The conclusion can also be derived from Axiom 4, but it really is just a consequence of Definition 5.)

Proof: Fix $k \in \mathscr{A}$. If $f(\omega)=g(\omega)$ for all $\omega\in E$, then $fEk=gEk$. Then $f\succ_E h$ iff $fEk \succ hEk$, hence iff $gEk\succ hEk$, i.e., iff $g\succ_E h$; the other statement is proved similarly. ■

Remark 4 (Negligible events) *Assume Axiom 5.*[5] *For all $E,N\in\Sigma$, N is negligible given E if and only if, for all $f,g\in\mathscr{A}$ with $f(\omega)\succ g(\omega)$ for $\omega\in E\setminus N$ implies $f\succ_E g$.*

That is, it is sufficient to restrict attention to states in F.

Proof: (If): assume that the property in the Remark holds. Consider f,g such that $f(\omega)\succ g(\omega)$ for all $\omega\notin N$. Then a fortiori this holds for all $\omega\in E\setminus N$, so by assumption $f\succ_E g$. Thus, N is negligible given E.

(Only if): assume that N is negligible given E. Consider f,g such that $f(\omega)\succ g(\omega)$ for all $\omega\in E\setminus N$. Fix $x,y\in X$ with $x\succ y$ (these exist by Axiom 5). Then $fEx(\omega)\succ gEy(\omega)$ for all $\omega\in(E\setminus N)\cup(\Omega\setminus E)$, hence a fortiori for all $\omega\notin N$. Since N is negligible given E, $fEx\succ_E gEy$. Then Observation 3 (Null Complement) implies that also $f\succ_E gEy$; apply Observation 3 again to conclude that $f\succ_E g$, so the property in the Remark holds. ■

Independence implies the following, standard dynamic-consistency property.

Next, I obtain a von Neumann-Morgenstern representation for (conditional) preferences over constant acts. Notice that the representation extends to preferences conditional upon events that do not belong to $\mathscr{F}$, but are supersets of some $F\in\mathscr{F}$.

Lemma 9 *Assume Axioms 1–6. Then there exists a cardinally unique, non-constant, affine function $u:X\to\mathbb{R}$ such that, for all $x,y\in X$, and all $E\in\Sigma$ such that $\succ_E$ is non-degenerate, $x\succ_E y$ iff $u(x)>u(y)$. In particular, this is the case for all E such that $E\supseteq F$ for some $F\in\mathscr{F}$.*

[5] It is actually sufficient to assume that $\succ=\succ_\Omega$ is non-degenerate.

Proof: Under the assumed axioms (in particular, taking $F = \Omega$ in Axioms 5 and 6), the restriction of $\succ$ to constant acts satisfies the von Neumann and Morgenstern (1947) axioms (see in particular Fishburn, 1970, Theorem 8.4); hence, there exists a cardinally unique, affine, non-constant $u : X \to \mathbb{R}$ such that $x \succ y$ iff $u(x) > u(y)$.

By Lemma 8, if $\succ_E$ is non-trivial, then $x \succ_E y$ iff $x \succ y$, and therefore iff $u(x) > u(y)$.

Now consider $E \in \Sigma$ and $F \in \mathscr{F}$ with $E \supseteq F$. By Axiom 5 (Non-degeneracy), there are $x', y' \in X$ with $x' \succ_F y'$. By Lemma 8, $x' \succ y'$. By Definition 5, $x'Fy' \succ y'$. By Irreflexivity (Axiom 1), not $y' \succ x'$, so Axiom 4 (Monotonicity) implies $x'Ey' \succ y'$. Hence, by Definition 5, $x' \succ_E y'$, i.e., $\succ_E$ is non-degenerate. Thus u represents $\succ_E$ on X as well. ■

The following result was established in the last step of the proof of Lemma 9, but it is worth emphasizing because it will be used several times below.

Corollary 3 *If $E, F \in \Sigma$, $\succ_F$ is non-degenerate, and $E \supset F$, then $\succ_E$ is non-degenerate. In particular, if $F \in \mathscr{F}$ and Axion 5 holds, then $\succ_E$ is non-degenerate.*

Observation 4 Assume Axioms 1–4 and 6. Then, for every $E \in \Sigma$ such that $\succ_E$ is non-degenerate, $\succ_E$ satisfies the following conditional versions of Axioms 1–4:

Strict Partial Order $\succ_E$ is irreflexive and transitive

Negative Transitivity for Prizes for all $x, y, z \in X$, not $y \succ_E x$ and not $z \succ_E y$ imply not $z \succ_E x$

Independence for all $f, g, h \in \mathscr{A}$, $f \succ_E g$ iff $\alpha f + (1-\alpha)h \succ_E \alpha g + (1-\alpha)h$

Monotonicity for all $x, y \in X$, $G \in \Sigma$, and $f, g \in \mathscr{A}$: if not $y \succ x$, then $yGf \succ g$ implies $xGf \succ_E g$ and $g \succ_E xEf$ implies $g \succ_E yEf$.

In addition, the following conditional version of Remark 3 holds:

Generalized Monotonicty For all $f, g, h \in \mathscr{A}$, if for all $\omega \in E$ not $g(\omega) \succ f(\omega)$ [resp. not $f(\omega) \succ g(\omega)$], then $g \succ_E h$ implies $f \succ_E h$ [resp. $h \succ_E g$ implies $h \succ_E f$]

(Recall also that, under the assumed axioms, $\succ$ and $\succ_E$ coincide on X by Lemma 8.)

Proof: Since under the noted axioms, Lemma 8 implies that $\succ$ and $\succ_E$ coincide on X, Strict Partial Order and Negative Transitivity for Prizes are immediate. Independence follows from Definition 5 and Axiom 3 for $\succ$. For Monotonicity, fix x, y with $x \succ y$ and assume $yGf \succ_E g$, i.e., $[yGf]Eh \succ gEh$ for some $h \in \mathscr{A}$. Equivalently, $y(G\cap E)[fEh] \succ gEh$. Then Axiom 4 implies $x(G\cap E)[fEh] \succ gEh$ [this holds trivially if $G \cap E = \emptyset$]. Equivalently, $[xGf]Eh \succ gEh$, so $xGf \succ_E g$, as required. The other implication is proved in the same way. Finally, for Generalized Monotonicity, suppose for all $\omega \in E$ not $g(\omega) \succ f(\omega)$. By Axiom 1, for any $k \in \mathscr{A}$ and $\omega \in \Omega \setminus E$, not $k(\omega) \succ k(\omega)$, so for all $\omega \in \Omega$ not $gEk(\omega) \succ fEk(\omega)$. Hence by Remark 3, $gEk \succ hEk$ implies $fEk \succ hEk$, i.e., $f \succ_E h$, as claimed. The other implication is proved similarly. ■

I now provide an equivalent condition for negligibility. The proof of sufficiency uses the notion of complementary acts from Siniscalchi (2009).

Lemma 10 *Assume Axioms 1–6. Fix events $F, N \in \Sigma$ and assume that $\succ_F$ is non-degenerate. Then N is negligible given F if and only if, for every $x, y, z \in X$ with $y \succ z$, $y \succ_F xNz$.*

Proof: Necessity is immediate. For sufficiency, consider $f, g \in \mathscr{A}$ such that $f(\omega) \succ g(\omega)$ for all $\omega \notin N$. Under the stated assumptions, $\succ$ and $\succ_F$ have a common EU representation on X by Lemma 9, with utility u. Let $y \in X$ be such that $u(y) = \frac{1}{2}\min_\omega u(f(\omega)) + \frac{1}{2}\max_\omega u(f(\omega))$, and let $\bar{f} \in \mathscr{A}$ be such that $u(\bar{f}(\omega)) = 2u(y) - u(f(\omega))$; such an act exists because, for every ω, $2u(y) - u(f(\omega)) = \min_{\omega'} u(f(\omega')) + \max_{\omega'} u(f(\omega')) - u(f(\omega)) \in \big(\min_\omega u(f(\omega)), \max_\omega u(f(\omega))\big)$. Therefore, for every ω, $u(y) = \frac{1}{2}u(f(\omega)) + \frac{1}{2}u(\bar{f}(\omega))$. By Independence (Axiom 3 and Observation 4), $f \succ_F g$ iff $\frac{1}{2}f + \frac{1}{2}\bar{f} \succ_F \frac{1}{2}g + \frac{1}{2}\bar{f}$. Since u represents $\succ$ on X, for every ω, not $y \succ \frac{1}{2}f(\omega) + \frac{1}{2}\bar{f}(\omega)$, so by Remark 3 and Observation 4 (Generalized Monotonicity), $y \succ_F \frac{1}{2}g + \frac{1}{2}\bar{f}$ implies $\frac{1}{2}f + \frac{1}{2}\bar{f} \succ_F \frac{1}{2}g + \frac{1}{2}\bar{f}$, and therefore it implies $f \succ_F g$. Hence, it is enough to show that $y \succ_F \frac{1}{2}g + \frac{1}{2}\bar{f}$.

Let $x, z \in X$ be such that $u(x) = \max_{\omega \in N} \frac{1}{2}u(g(\omega)) + \frac{1}{2}u(\bar{f}(\omega))$ and $u(z) = \max_{\omega \notin N} \frac{1}{2}u(g(\omega)) + \frac{1}{2}u(\bar{f}(\omega))$. Note that $\frac{1}{2}u(g(\omega)) + \frac{1}{2}u(\bar{f}(\omega)) = \frac{1}{2}u(g(\omega)) + u(y) - \frac{1}{2}u(f(\omega)) = u(y) - \frac{1}{2}[u(f(\omega)) - u(g(\omega))]$; since $f(\omega) \succ g(\omega)$ for all $\omega \notin N$, and u represents $\succ$ on X, $\frac{1}{2}u(g(\omega)) + \frac{1}{2}u(\bar{f}(\omega)) < u(y)$ for every $\omega \notin N$, so in particular $y \succ z$ (recall that attention is restricted to simple acts). Hence, the condition in the Remark implies $y \succ_F xNz$. But since u represents $\succ$ on X, for every ω, not $[\frac{1}{2}g + \frac{1}{2}\bar{f}](\omega) \succ xFz(\omega)$. Hence, by Remark 3 and Observation 4 (Generalized Monotonicity), $y \succ_F \frac{1}{2}g + \frac{1}{2}\bar{f}$. ■

Corollary 4 *For every $E \in \Sigma$ such that $F \subseteq E$ for some $F \in \mathscr{F}$, $\Omega \setminus E$ is negligible given E.*

Proof: Fix x, y, z with $y \succ z$. Note that $x(\Omega \setminus E)z = zFx$. Hence $\Omega \setminus F$ is negligible given E iff $y \succ_E zEx$. By Observation 3, this is equivalent to $y \succ_E z$. But by Lemma 9, this holds iff $y \succ z$. ■

Results concerning negligible events.

Lemma 11 *Assume Axioms 1–6. Fix $F, G, N, M \in \Sigma$ such that $\succ_F$ and $\succ_G$ are non-degenerate.*

1. *If N is negligible given F and $M \subset N$, then M is negligible given F. Thus, if M is not negligible given F, neither is N.*

2. *If $N \subseteq F$ is negligible given F, and $F \subseteq G$, then N is negligible given G. Thus, if N is not negligible given G, neither is it negligible given F.*

3. *If N and M are both negligible given F, then so is $N \cup M$.*

4. *If N is negligible given G, $G \supset F \supset N$, and $G \setminus F$ is negligible given G, then N is also negligible given F.*

Proof: Since $\succ_F$ and $\succ_G$ are non-degenerate, they agree with $\succ$ on X, and are represented by the utility function u, per Lemma 9. These facts will be used throughout the proof.

1: if $f(\omega) \succ g(\omega)$ for all $\omega \notin M$, then a fortiori this holds for all $\omega \notin N \supset M$; since N is negligible given F, $f \succ_F g$. Since this is the case for all $f, g \in \mathscr{A}$, M is negligible given F.

2: if $f(\omega) \succ g(\omega)$ for all $\omega \notin N$, then $f \succ_F g$ because N is negligible given F. By the definition of conditional preferences, in particular $fFg \succ g$. Since $f(\omega) \succ g(\omega)$ for all $\omega \in G \setminus F$ and $G \supseteq F \supseteq N$, by Irreflexivity (Axiom 1) for all ω not $fFg(\omega) \succ fGg(\omega)$. Then, by Remark 3 (Generalized Monotonicity) $fGg \succ g$. But then, by the definition of conditional preferences, $f \succ_G g$. Since f, g were arbitrary, N is negligible for G.

3: Fix F, N, M as in the statement. It is enough to consider the case $N \cap M = \emptyset$: this is because (i) for arbitrary negligible events N, M, $N \cup M = (N \setminus M) \cup (N \cap M) \cup (M \setminus N)$, and the sets on the rhs are all negligible by part 1; and (ii) if the union of two disjoint negligible events is negligible, by induction so is the union of finitely many (in particular, three) disjoint negligible events.

Since $\succ$ satisfies Axioms 1–6 and $\succ_F, \succ_G$ are non-degenerate, one can invoke the alternative characterization of negligibility in Lemma 10; also, the restriction of $\succ$ to X is represented by the non-constant, affine function $u : X \to \mathbb{R}$. It is without loss of generality to assume that $[-1, 1] \subseteq u(X)$; also, let $z_0 \in X$ be such that $u(z_0) = 0$.

For arbitrary $x, y, z \in X$, there exists $\alpha \in (0, 1)$ such that $u \circ [\alpha x + (1-\alpha) z_0] = \alpha u(x) \in [-\frac{1}{2}, \frac{1}{2}]$, and similarly $u \circ [\alpha y + (1-\alpha) z_0], u \circ [\alpha z + (1-\alpha) z_0] \in [-\frac{1}{2}, \frac{1}{2}]$. Then, by Independence (Axiom 3), $y \succ z$ iff $\alpha y + (1-\alpha) z_0 \succ \alpha z + (1-\alpha) z_0$, and $y \succ_F xNy$ iff $\alpha y + (1-\alpha) z_0 \succ_F [\alpha x + (1-\alpha) z_0] N [\alpha z + (1-\alpha) z_0]$. Thus, it is enough to show that $M \cup N$ satisfies the condition of Lemma 10 for $x, y, z \in X$ such that $u(x), u(y), u(z) \in [-\frac{1}{2}, \frac{1}{2}]$.

Furthermore, for any such tuple $x, y, z \in X$, there is $\bar{z} \in X$ such that $u(\bar{z}) = -u(z)$, so that $u \circ [\frac{1}{2} z + \frac{1}{2} \bar{z}] = 0 = u(z_0)$, so neither $\frac{1}{2} z + \frac{1}{2} \bar{z} \succ z_0$ nor $z_0 \succ \frac{1}{2} z + \frac{1}{2} \bar{z}$. By Independence (Axiom 3), $y \succ z$ iff $\frac{1}{2} y + \frac{1}{2} \bar{z} \succ \frac{1}{2} z + \frac{1}{2} \bar{z}$, hence (from the EU representation of $\succ$ on X) iff $\frac{1}{2} y + \frac{1}{2} \bar{z} \succ z_0$. Independence (Axiom 3 and Observation 4) also implies that $y \succ_F xNz$ iff $\frac{1}{2} y + \frac{1}{2} \bar{z} \succ_F (\frac{1}{2} x + \frac{1}{2} \bar{z}) N (\frac{1}{2} z + \frac{1}{2} \bar{z})$. Since not $\frac{1}{2} z + \frac{1}{2} \bar{z} \succ z_0$ and not $z_0 \succ \frac{1}{2} z + \frac{1}{2} \bar{z}$, Monotonicity (Axiom 4 and

Observation 4) imply that, if $\frac{1}{2}y+\frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x+\frac{1}{2}\bar{z})N(\frac{1}{2}z+\frac{1}{2}\bar{z})$, then $\frac{1}{2}y+\frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x+\frac{1}{2}\bar{z})Nz_0$, and conversely, if $\frac{1}{2}y+\frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x+\frac{1}{2}\bar{z})Nz_0$, then $\frac{1}{2}y+\frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x+\frac{1}{2}\bar{z})N(\frac{1}{2}z+\frac{1}{2}\bar{z})$. Therefore, $y \succ_F xNz$ iff $\frac{1}{2}y+\frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x+\frac{1}{2}\bar{z})Nz_0$. Clearly, $u\circ[\frac{1}{2}y+\frac{1}{2}\bar{z}], u\circ[\frac{1}{2}x+\frac{1}{2}\bar{z}] \in [-\frac{1}{2},\frac{1}{2}]$. Thus, it is enough to show that, for all $x,y \in X$ with $u(x), u(y) \in [-\frac{1}{2},\frac{1}{2}]$, $y \succ z_0$ implies $y \succ_F x[M\cup N]z_0$.

Fix such $x,y \in X$. Suppose that $y \succ z_0$. Since $u(x) \in [-\frac{1}{2},\frac{1}{2}] \subset [-1,1] \subseteq u(X)$, there is $x' \in X$ such that $u(x') = 2u(x)$. Since both N and M are negligible given F, $y \succ z_0$ implies both $y \succ_F x'Nz_0$ and $y \succ_F x'Mz_0$. Therefore, by Independence (Axiom 3 and Observation 4)[6], $y \succ_F (\frac{1}{2}x'+\frac{1}{2}z_0)(N\cup M)z_0$. But $u\circ[\frac{1}{2}x'+\frac{1}{2}z_0] = \frac{1}{2}u(x')+\frac{1}{2}u(z_0) = \frac{1}{2}\cdot 2u(x) = u(x)$, so by the EU representation of $\succ$ on X, not $x \succ \frac{1}{2}x'+\frac{1}{2}z_0$. Therefore, by Monotonicity (Axiom 4 and Observation 4), $y \succ_F x(N\cup M)z_0$. This proves the claim.

4: Since N and $G\setminus F$ are both negligible given G, by Lemma 11 part 3 so is $N\cup(G\setminus F)$. Therefore, for all $x,y,z \in X$ with $y \succ z$, $y \succ_G x[N\cup(G\setminus F)]z$.

If not $y \succ x$, then by Axiom 4 (Monotonicity) $y \succ_G x[N\cup(G\setminus F)]z$ implies $y \succ_G xNy(G\setminus F)z$. By the definition of conditional preferences, $yGz \succ xNy(G\setminus F)z$. Observe that both acts assign the prize y to states in $G\setminus F$, and the prize z to states not in G; hence, by the definition of conditional preferences, $y \succ_F xNz$.

If instead $y \succ x$, then $y \succ_F x$ as well. There are two subcases. If $x \succ_F z$, then not $z \succ_F x$ by Irreflexivity, and so Axiom 4 (Monotonicity) implies that $y \succ_F xNz$. If instead not $x \succ_F z$, then $y \succ_F z$ and Axiom 4 again imply $y \succ_F xNz$. This completes the proof. ■

Corollary 5 *For any $F,N \in \Sigma$, if $\succ_F$ is non-degenerate and N is negligible given F, then $F\setminus N$ is not negligible given F.*

Proof: If N is negligible for F, so is $F\cap N$ by Part 1. Suppose $F\setminus N$ is also negligible for F. Then $F=(F\cap N)\cup(F\setminus N)$ is negligible given F by Part 3. Using the characterization of negligi-

[6] Suppose that $f \succ g$ and $f' \succ g'$. Then Independence implies $\frac{1}{2}f+\frac{1}{2}f' \succ \frac{1}{2}g+\frac{1}{2}f'$ and $\frac{1}{2}f'+\frac{1}{2}g \succ \frac{1}{2}g'+\frac{1}{2}g$. Thus, by Transitivity, $\frac{1}{2}f+\frac{1}{2}f' \succ \frac{1}{2}g+\frac{1}{2}g'$.

ble events in Lemma 10, choose $x, y, z \in X$ with $x \succ y \succ z$; then $y \succ_F xFz$. By the definition of conditional preferences, in particular $yFz \succ (xFz)Fz = xFz$, i.e., again by Definition 5, $y \succ_F x$. But by Axiom 5 and Lemma 8, $x \succ_F y$, which contradicts the fact that $\succ_F$ is irreflexive (Axiom 1 and Observation 4). ■

Recall that, for every $s_{-i} \in S_{-i}$, $[s_{-i}] = \{s_{-i}\} \times W$.

Corollary 6 *For any $E \in \Sigma$, if $\succ_E$ is non-degenerate, then $\sigma(E)$ is not negligible given E, and $E \setminus \sigma(E)$ is negligible given E.*

Proof: Let $N \equiv \bigcup\{[s_{-i}] : [s_{-i}]$ is negligible given $E\}$. By part 3 of the Lemma, N is negligible given E. By definition, $\sigma(E) = E \setminus N$. Since $E \setminus \sigma(E) \subseteq N$,[7] by part 1 of the Lemma it is also negligible given E. Finally, since N is negligible given E, by Corollary 5, $\sigma(E) = E \setminus N$ is not negligible given E. ■

D Sufficiency: main argument

Lemma 12 *Fix non-empty $E, F \in \Sigma$ and $p, q \in \Delta(\Sigma)$ such that $p(E) = q(F) = 1$, $p(E \cap F) > 0$, $q(E \cap F) > 0$, and $p(\cdot|E \cap F) = q(\cdot|E \cap F)$. Then:*

1. *there is a unique $r \in \Delta(\Sigma)$ such that $r(E \cup F) = 1$, $r(E) > 0$, $r(F) > 0$, $r(\cdot|E) = p$, and $r(\cdot|F) = q$.*

[7] In general, N need not be a subset of E. Hence we allow for inclusion rather than equality. However, if $E = (\text{proj}_{S_{-i}} E) \times W$, as is the case for unions of elements from $\mathscr{F}$, one obtains an equality.

2. *for every simple, Σ-measurable $\phi:\Omega\to\mathbb{R}$, the function $\phi':\Omega\to\mathbb{R}$ such that*

$$\phi'(\omega)=\begin{cases}0 & \omega\in\Omega\setminus F\\ \frac{\mathrm{E}_p\phi}{p(E\cap F)} & \omega\in E\cap F\\ \phi(\omega) & \omega\in F\setminus E\end{cases}$$

satisfies $\mathrm{E}_p\phi'=\mathrm{E}_p\phi$ *and* $\mathrm{E}_r\phi=r(F)\cdot\mathrm{E}_q\phi'$.

Proof: (1): let $\rho\equiv\frac{p(E\cap F)}{q(E\cap F)}$ and define r by

$$r(G)=\frac{p(G\cap E)+\rho\cdot q(G\cap[F\setminus E])}{1+\rho\cdot q(F\setminus E)}\quad\forall G\in\Sigma. \tag{7}$$

In particular, since $p(E)=1$,

$$r(E\cup F)=1,\quad r(E)=\frac{1}{1+\rho q(F\setminus E)}>0,\quad\text{and}\quad r(F)=\frac{p(E\cap F)+\rho\cdot q(F\setminus E))}{1+\rho q(F\setminus E)}>0.$$

It is immediate that, for $G\subseteq E$, $r(G|E)=p(E)$. In particular, this is true for $G\subseteq E\cap F$; since for such G by assumption $p(G|E\cap F)=q(G|E\cap F)$, it is also the case that

$$r(G|F)=\frac{r(G)}{r(F)}=\frac{r(G)}{r(E)}\cdot\frac{r(E)}{r(F)}=p(G)\cdot\frac{r(E)}{r(F)}=p(G)\cdot\frac{1}{p(E\cap F)+\rho q(F\setminus E)}=$$
$$=\frac{p(G)}{p(E\cap F)}\cdot\frac{1}{1+\frac{q(F\setminus E)}{q(E\cap F)}}=\frac{q(G)}{q(E\cap F)}\cdot\frac{q(E\cap F)}{q(E\cap F)+q(F\setminus E)}=q(G)$$

for $G\subseteq E\cap F$.

Finally, consider $G\subseteq F\setminus E$. We have

$$r(G|F)=\frac{\rho\cdot q(G)}{p(E\cap F)+\rho\cdot q(F\setminus E)}=\frac{\rho\cdot q(G)}{\rho\cdot q(E\cap F)+\rho\cdot q(F\setminus E)}=q(G),$$

as required.

Now let $r'\in\Delta(\Sigma)$ be such that $r'(E\cup F)=1$, $r'(E)>0$, $r'(F)>0$, $r'(\cdot|E)=p$, and $r'(\cdot|F)=q$. Then

$$\frac{r'(E\setminus F)}{r'(E\cap F)}=\frac{r'(E\setminus F|E)}{r'(E\cap F|E)}=\frac{p(E\setminus F)}{p(E\cap F)}$$

and similarly $\frac{r'(F\setminus E)}{r'(E\cap F)}=\frac{q(F\setminus E)}{q(E\cap F)}$. Therefore,

$$1=r'(E\cap F)+r'(E\setminus F)+r'(F\setminus E)=r'(E\cap F)\left(1+\frac{p(E\setminus F)}{p(E\cap F)}+\frac{q(F\setminus E)}{q(E\cap F)}\right).$$

Thus, the value of $r'(E\cap F)$ is determined by p and q, and hence must coincide with $r(E\cap F)$ [this can also be verified directly by taking $G=E\cap F$ in Eq. (7)]. Hence, also $r'(E\setminus F)=r(E\setminus F)$ and $r'(F\setminus E)=r(F\setminus F)$. Thus $r'(E)=r(E)$ and $r'(F)=r(F)$. If $G\in\Sigma$ satisfies $G\subseteq E$, then $r'(G)=r'(G|E)\cdot r'(E)=p(G)\cdot r(E)=r(G|E)\cdot r(E)=r(G)$; similarly, if $G\subseteq F$, the $r'(G)=r(G)$. Finally, for arbitrary $G\in\Sigma$, $r'(G)=r'(G\cap[E\cap F])=r'(G\cap E)+r'(G\cap[F\setminus E])=r(G\cap E)+r(G\cap [F\setminus E])=r(G)$.

(2): by direct calculation,

$$\mathrm{E}_p\phi'=p(E\setminus F)\cdot 0+p(E\cap F)\cdot\frac{E_p\phi}{p(E\cap F)}=E_p\phi.$$

Moreover, $q=r(\cdot|F)$, so $r(F)\cdot q(G)=r(F)\cdot r(G|F)=r(G)$ for all $G\subseteq F$, so

$$r(F)\cdot E_q\phi'=r(F)\cdot q(E\cap F)\cdot\frac{E_p\phi}{p(E\cap F)}+r(F)\cdot\int_{F\setminus E}\phi\,dq=r(E\cap F)\cdot\frac{E_p\phi}{p(E\cap F)}+\int_{F\setminus E}\phi\,dr$$

and finally, since $p=r(\cdot|E)$, and in particular $p(E\cap F)=r(E\cap F|E)=\frac{r(E\cap F)}{r(E)}$, the rhs equals

$$r(E)\cdot E_p\phi+\int_{F\setminus E}\phi\,dr=r(E)\int\phi\,dr(\cdot|E)+\int_{F\setminus E}\phi\,dr=\mathrm{E}_r\phi=\int_E\phi\,dr+\int_{F\setminus E}\phi\,dr=\mathrm{E}_r\phi.$$

■

Lemma 13 *Assume Axioms 1–6. Let u be as in Lemma 9. If there are $p,q\in\Delta(\Sigma)$ with $p(E)=q(F)=1$, $p(E\cap F)>0$, and $q(E\cap F)>0$ such that*

$$\mathrm{E}_p u\circ f>\mathrm{E}_p u\circ g\Rightarrow f\succ_E g \tag{8}$$

$$\mathrm{E}_q u\circ f>\mathrm{E}_q u\circ g\Rightarrow f\succ_F g \tag{9}$$

then $p(\cdot|E\cap F) = q(\cdot|E\cap F)$*, and*

$$\mathrm{E}_r\, u\circ f > \mathrm{E}_r\, u\circ g \Rightarrow f \succ_{E\cup F} g, \tag{10}$$

where r *is as in Lemma 12.*

Notation: in this proof, it is necessary to consider composites of three acts. For $E_1, E_2 \in \Sigma$ such that $E_1 \cap E_2 = \emptyset$ and $f_1, f_2, f_3 \in \mathscr{A}$, let $f_1 E_1 f_2 E_2 f_3$ denote the act g such that $g(\omega) = f_1(\omega)$ for $\omega \in E_1$, $g(\omega) = f_2(\omega)$ for $\omega \in E_2$, and $g(\omega) = f_3(\omega)$ for $\omega \notin E_1 \cup E_2$.

Proof: Since u is non-constant, $\succ_E$ and $\succ_F$ are non-degenerate; hence, by Corollary 3, so is $\succ_{E\cup F}$.

Consider $D \in \Sigma$ such that $D \subseteq E\cap F$. Fix $x, y \in X$ with $u(x) > u(y)$, so $x \succ_E y$ and $x \succ_F y$. Fix $\alpha \in [0,1]$. Suppose that $p(D|E\cap F) > \alpha$, or $p(D) > P(E\cap F)\alpha$; then $u(y) + p(D)[u(x) - u(y)] > u(y) + p(E\cap F)\alpha[u(x) - u(y)]$ because $u(x) > u(y)$. Equivalently, $p(D)u(x) + [1 - p(D)]u(y) > p(E\cap F)[\alpha u(x) + (1-\alpha)u(y)] + [1 - p(E\cap F)]u(y)$. By Eq. (8), this implies $x D y \succ_E \{\alpha u(x) + (1-\alpha)u(y)\}(E\cap F)y$. By Definition 5 (which is well-posed by Remark 1), since $D \subseteq E\cap F \subseteq E$, in particular $x D y \succ \{\alpha u(x) + (1-\alpha)u(y)\}(E\cap F)y$. Similarly, $p(D|E\cap F) < \alpha$ implies $x D y \prec \{\alpha u(x) + (1-\alpha)u(y)\}(E\cap F)y$. By a symmetric argument, using Eq. (9) and the fact that $D \subseteq E\cap F \subseteq F$, $q(D|E\cap F) > \alpha$ implies $x D y \succ \{\alpha u(x) + (1-\alpha)u(y)\}(E\cap F)y$ and $q(D|E\cap F) < \alpha$ implies $x D y \prec \{\alpha u(x) + (1-\alpha)u(y)\}(E\cap F)y$.

Suppose that $q(D|E\cap F) > p(D|E\cap F)$, and fix $\alpha \in \big(p(D|E\cap F), q(D|E\cap F)\big)$. Since $q(D|E\cap F) > \alpha$, $x D y \succ \{\alpha u(x) + (1-\alpha)u(y)\}(E\cap F)y$; but since $p(D|E\cap F) < \alpha$, $x D y \prec \{\alpha u(x) + (1-\alpha)u(y)\}(E\cap F)y$, which contradicts Irreflexivity in Axiom 1. Similarly, $q(D|E\cap F) < p(D|E\cap F)$ also leads to a contradiction. Hence, $q(D|E\cap F) = p(D|E\cap F)$.

Now assume wlog that $u(X) \supseteq [-1,1]$. Consider first acts $f, g \in \mathscr{A}$ such that $u(f(\omega))/p(E\cap F) \in (-1,1)$ and $u(g(\omega))/p(E\cap F) \in (-1,1)$ for every ω. Suppose that $\mathrm{E}_r\, u\circ f > \mathrm{E}_r\, u\circ g$. Then, since f, g take on finitely many values, there exists $\epsilon > 0$ such that

$$\frac{u(f(\omega)) - \epsilon}{p(E\cap F)} \in (-1,1)\ \forall\omega, \quad \frac{u(g(\omega)) + \epsilon}{p(E\cap F)} \in (-1,1)\ \forall\omega, \quad \text{and} \quad \mathrm{E}_r\, u\circ f - \epsilon > \mathrm{E}_r\, u\circ g + \epsilon. \tag{11}$$

Therefore, there exist $x, y \in X$ such that $u(x) = [\mathrm{E}_p u \circ f - \epsilon]/p(E \cap F)$ and $u(y) = [\mathrm{E}_p u \circ g + \epsilon]/p(E \cap F)$; also let $z \in X$ be such that $u(z) = 0$. Moreover, since $u(f(\omega)) - \epsilon \in \big(-p(E \cap F), p(E \cap F)\big) \subseteq (-1,1)$ and $u(g(\omega)) + \epsilon \in \big(-p(E \cap F), p(E \cap F)\big) \subseteq (-1,1)$, there exist $f^{-\epsilon}, g^{+\epsilon} \in \mathscr{A}$ such that $u(f^{-\epsilon}(\omega)) = u(f(\omega) - \epsilon$ and $u(g^{+\epsilon}(\omega)) = u(g(\omega) + \epsilon$ for all ω. Then $u(x) = \mathrm{E}_p u \circ f^{-\epsilon}/p(E \cap F)$ and $u(y) = \mathrm{E}_p u \circ g^{+\epsilon}/p(E \cap F)$; furthermore, $\mathrm{E}_r u \circ f^{-\epsilon} = \mathrm{E}_r u \circ f - \epsilon > \mathrm{E}_r u \circ g + \epsilon = \mathrm{E}_r u \circ g^{+\epsilon}$.

Consider the acts $f^{-\epsilon}[F \setminus E]x[E \cap F]z$ and $g^{+\epsilon}[F \setminus E]y[E \cap F]z$. By part 2 of Lemma 12,

$$\mathrm{E}_q u \circ \big(f^{-\epsilon}[F \setminus E]x[E \cap F]z\big) = \mathrm{E}_r u \circ f^{-\epsilon} > \mathrm{E}_r u \circ g^{+\epsilon} = \mathrm{E}_q u \circ \big(g^{+\epsilon}[F \setminus E]y[E \cap F]z\big),$$

so Eq. (9) implies that

$$f^{-\epsilon}[F \setminus E]x[E \cap F]z \succ_F g^{+\epsilon}[F \setminus E]y[E \cap F]z.$$

By Definition 5, taking z as the act on the complement of F,

$$f^{-\epsilon}[F \setminus E]x[E \cap F]z \succ g^{+\epsilon}[F \setminus E]y[E \cap F]z;$$

and applying Definition 5 again, taking z as the act on the complement of $E \cup F$, these preference rankings imply

$$f^{-\epsilon}[F \setminus E]x[E \cap F]z \succ_{E \cup F} g^{+\epsilon}[F \setminus E]y[E \cap F]z. \tag{12}$$

Lemma 12 part 2 also implies that $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ f^{-\epsilon} = u(x)p(E \cap F) = \mathrm{E}_p u \circ x[E \cap F]z$. By Eq. (8), $f \succ_E x[E \cap F]z$. Similarly, $y[E \cap F]z \succ_E g$. By Definition 5, taking $f^{-\epsilon}[F \setminus E]z$ and, respectively, $g^{+\epsilon}[F \setminus E]z$ as the act on the complement of E,

$$f^{-\epsilon}[F \setminus E]fEz \succ f^{-\epsilon}[F \setminus E]x[E \cap F]z \quad \text{and} \quad g^{+\epsilon}[F \setminus E]y[E \cap F]z \succ g^{+\epsilon}[F \setminus E]gEz;$$

and again by Definition 5, taking z as the act on the complement of $E \cup F$,

$$f^{-\epsilon}[F \setminus E]f \succ_{E \cup F} f^{-\epsilon}[F \setminus E]x[E \cap F]z \quad \text{and} \quad g^{+\epsilon}[F \setminus E]y[E \cap F]z \succ_{E \cup F} g^{+\epsilon}[F \setminus E]g. \tag{13}$$

Combining Eqs. (12) and (13), by Transitivity (Observation 4)

$$f^{-\epsilon}[F \setminus E]f \succ_{E \cup F} g^{+\epsilon}[F \setminus E]g.$$

Finally, for every $\omega \in F \setminus E$, $u\big(f^{-\epsilon}[F \setminus E]f\big)(\omega) = u(f(\omega)) - \epsilon < u(f(\omega))$ and $u\big(g^{+\epsilon}[F \setminus E]g\big)(\omega) = u(g(\omega)) + \epsilon > u(g(\omega))$; and for all $\omega \notin F \setminus E$, $u\big(f^{-\epsilon}[F \setminus E]f\big)(\omega) = u(f(\omega))$ and $u\big(g^{+\epsilon}[F \setminus E]g\big)(\omega) = u(g(\omega))$. Thus, since u represents $\succ$ on X, for all ω, not $\big(f^{-\epsilon}[F \setminus E]f\big)(\omega) \succ f(\omega)$ and not $g(\omega) \succ \big(g^{+\epsilon}[F \setminus E]g\big)(\omega)$. Hence, by Generalized Monotonicity (Observation 4), $f \succ_{E \cup F} g$.

Finally, consider arbitrary $f, g \in \mathscr{A}$, and suppose that $\mathrm{E}_r u \circ f > \mathrm{E}_r u \circ g$. By finiteness there are $z', z'' \in X$ such that $z' \succ f(s) \succ z''$ and $z' \succ g(s) \succ z''$ for all s. Let $\alpha \in (0,1)$ be such that $\alpha u(z')/p(E \cap F), \alpha u(z'')/p(E \cap F) \in (-1,1)$, and consider the acts $f' = \alpha f + (1-\alpha)z$ and $g' = \alpha g + (1-\alpha)z$, where again $u(z) = 0$. Then $u(f'(\omega))/p(E \cap F)), u(g'(\omega))/p(E \cap F) \in (-1,1)$ for all ω, and furthermore $\mathrm{E}_r u \circ f' = \alpha \mathrm{E}_r u \circ f > \alpha \mathrm{E}_r u \circ g = \mathrm{E}_r u \circ g'$, so by the argument just given, $f' \succ_{E \cup F} g'$, i.e., $\alpha f + (1-\alpha)z \succ_{E \cup F} \alpha g + (1-\alpha)z$. By Independence (Observation 4), this implies $f \succ_{E \cup F} g$. ■

Lemma 14 *Assume Axioms 1–6. Let u be as in Lemma 9. If there are $p, q \in \Delta(\Sigma)$ with $p(E) = q(F) = 1$, $p(E \cap F) = 0$, and $q(E \cap F) > 0$ such that*

$$\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g \Rightarrow f \succ_E g \tag{14}$$

$$\mathrm{E}_q u \circ f > \mathrm{E}_q u \circ g \Leftrightarrow f \succ_F g \tag{15}$$

then $F \setminus E$ is negligible given $E \cup F$, and

$$\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g \Rightarrow f \succ_{E \cup F} g. \tag{16}$$

Proof: Since u is non-constant, $\succ_E$ and $\succ_F$ are non-degenerate; hence, by Corollary 3, so is $\succ_{E \cup F}$.

Consider $x, y, z \in X$ with $y \succ z$. Assume wlog that $u(z) = 0$. Since $p(E \cap F) = 0$, $\mathrm{E}_p u \circ x[E \cap F]z = u(z) < u(y)$, so by Eq. (14) $y \succ_E x[E \cap F]z$. Since this holds for all x, y, z with $y \succ z$, $E \cap F$ is negligible given E by Lemma 10.

Now suppose first that $\sup u(X) > \frac{1-q(E\cap F)}{q(E\cap F)} u(x)$. Then by definition there is $\bar{x} \in X$ such that $u(\bar{x}) > \frac{1-q(E\cap F)}{q(E\cap F)} u(x)$. By the argument just given, in particular $y \succ_E \bar{x}[E \cap F]z$. By Definition 5 taking $y[F \setminus E]z$ as the act on the complement of E,

$$y[E \cup F]z \succ \bar{x}[E \cap F]y[F \setminus E]z.$$

Since $y \succ z$, Asymmetry (implied by Axiom 1) yields not $z \succ y$, and furthermore not $y \succ y$ by Irreflexivity (Axiom 1), so for every ω, not $\big(\bar{x}[E \cap F]z\big)(\omega) \succ \big(\bar{x}[E \cap F]y[F \setminus E]z\big)(\omega)$. Then, by Generalized Monotonicity (Remark 3),

$$y[E \cup F]z \succ \bar{x}[E \cap F]z$$

and therefore, by Definition 5, taking z as the act on the complement of $E \cup F$,

$$y \succ_{E \cup F} \bar{x}[E \cap F]z.$$

By the choice of $\bar{x}$, $\mathrm{E}_q\, u \circ \bar{x}[E \cap F]z = q(E \cap F)u(\bar{x}) > q(E \cap F) \cdot \frac{1-q(E \cap F)}{q(E \cap F)} u(x) = [1 - q(E \cap F)u(x) = \mathrm{E}_q\, u \circ x[F \setminus E]z$, so by Eq. (15),

$$\bar{x}[E \cap F]z \succ_F x[F \setminus E]z.$$

Applying Definition 5 with z as the act on the complement of F,

$$\bar{x}[E \cap F]z \succ x[F \setminus E]z$$

and applying it again with z as the act on the complement of $E \cup F$,

$$\bar{x}[E \cap F]z \succ_{E \cup F} x[F \setminus E]z.$$

Thus, by Transitivity (Observation 4), $y \succ_{E \cup F} x[F \setminus E]z$.

Finally, if $\frac{1-q(E \cap F)}{q(E \cap F)} u(x) \geq \sup u(X)$, note that $\sup u(X) \geq u(y) > u(z) = 0$. For α sufficiently small, $x' = \alpha x + (1-\alpha)z$ and $y' = \alpha y + (1-\alpha)z$ satisfy $y' \succ z$ by Independence and $\sup u(X) > \frac{1-q(E \cap F)}{q(E \cap F)} u(x')$. By the result just shown, $y' \succ_{E \cup F} x'[F \setminus E]z$, i.e., $\alpha y + (1-\alpha)z \succ_{E \cup F} \alpha\big(x[F \setminus E]z\big) + (1-\alpha)z$. By Independence (Observation 4), $y \succ_{E \cup F} x[F \setminus E]z$.

Since $x, y, z \in X$ were arbitrary prizes with $y \succ z$, $F \setminus E$ is negligible given $E \cup F$. Furthermore, $E \cap F$ is negligible given E, and hence also given $E \cup F$ by Lemma 11 part 2. Therefore, $F = (E \cap F) \cup (F \setminus E)$ is negligible given $E \cup F$ by part 3 of the same Lemma.

Now consider $f, g \in \mathscr{A}$ with $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g$. Let $x, y \in X$ be such that $\mathrm{E}_p u \circ f > u(x) > u(y) > \mathrm{E}_p u \circ g$. Since u represents $\succ$ on X, $x \succ y$. Since $F \setminus E$ is negligible given $E \cup F$,

$$f[F \setminus E]x \succ_{E \cup F} g[F \setminus E]y.$$

Finally, since $\mathrm{E}_p u \circ f > u(x)$, by Eq. (14) $f \succ_E x$. With f as the act on the complement of E in Definition 5, $f \succ xEf$, and again by taking f as the act on the complement of $E \cup F$, $f \succ_{E \cup F} xEf$. Finally, for all $\omega \in E \cup F$, $(xEf)(\omega) = (f[F \setminus E]x)(\omega)$, so by Observation 3 $f \succ_{E \cup F} f[F \setminus E]x$. Similarly, $g[F \setminus E]y \succ_{E \cup F} g$. By Transitivity (Observation 4), $f \succ_{E \cup F} g$. ■

Remark 5 *Fix $E \in \Sigma$ such that $\succ_E$ is irreflexive, a non-constant, affine $u : X \to \mathbb{R}$, and $p, q \in \Delta(\Sigma)$ with $p(E) = 1$. If, for all $f, g \in \mathscr{A}$, both $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g \Rightarrow f \succ_E g$ and $\mathrm{E}_q u \circ f > \mathrm{E}_q u \circ g \Rightarrow f \succ_E g$, then $p = q$.*

Proof: Consider $D \in \Sigma$ with $D \subseteq E$. Fix $x, y \in X$ with $u(x) > u(y)$; wlog assume $u(x) = 1$ and $u(y) = 0$. Suppose $\alpha \in \big(p(D), 1\big]$. Then $\mathrm{E}_p u \circ xDy = p(D) < \alpha = u(\alpha x + (1-\alpha)y)$, so $xDy \prec_E \alpha x + (1-\alpha)y$. If $\mathrm{E}_q u \circ xDy > u(\alpha x + (1-\alpha)y)$, then $xDy \succ_E \alpha x + (1-\alpha)y$, which violates Irreflexivity; thus, $q(D) = \mathrm{E}_q u \circ xDy \le u(\alpha x + (1-\alpha)y) = \alpha$. Similarly, $\alpha \in \big[0, p(D)\big)$ implies $q(D) \ge \alpha$. Now suppose $p(D) < q(D)$ and choose $\alpha \in \big(p(D), q(D)\big)$: then $\alpha > p(D)$ implies $\alpha \ge q(D)$, contradiction. Similarly, if $p(D) > q(D)$, choose $\alpha \in \big(q(D), p(D)\big)$: then $\alpha < p(D)$ implies $\alpha \le q(D)$, contradiction. Thus, $p(D) = q(D)$. ■

Basic EU representation for conditioning events:

Lemma 15 *Assume Axioms 1–8. Fix $F \in \mathscr{F}$. Then:*

1. *$\succ_F$ and $\succ_{\sigma(F)}$ are non-degenerate.*

2. *$\succ_{\sigma(F)}$ admits an EU representation (u,p), where $p \in \Delta(\Sigma)$ satisfies $p(\sigma(F)) = 1$ and $u : X \to \mathbb{R}$ is the function that represents $\succ$ on X per Lemma 9.*

3. *for every $f,g \in \mathscr{A}$, $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g$ implies $f \succ_F g$*

4. *an event $N \in \Sigma$ is negligible given F if and only if $p(N) = 0$.*

Proof:

1: $\succ_F$ is non-degenerate by Corollary 3. Moreover, by Corollary 6, $F \setminus \sigma(F)$ is negligible given F, and $x\sigma(F)y(\omega) = x \succ y$ for all $\omega \in F \setminus [F \setminus \sigma(F)] = \sigma(F)$, so $x\sigma(F)y \succ_F y$ by Remark 4. By the definition of conditional prefeences, in particular $x\sigma(F)y \succ y$, and so, again by Definition 5, $x \succ_{\sigma(F)} y$. Thus, $\succ_{\sigma(F)}$ is also non-degenerate.

2: The relation $\succ_{\sigma(F)}$ is irreflexive by Axiom 1 and Observation 4, and negatively transitive by Axiom 7. It satisfies Independence by Axiom 3 and Observation 4; and it satisfies the Archimedean property by Axiom 8. As was just noted, $x \succ_{\sigma(F)} y$, so the preference is non-degenerate. Finally, Observation 2 shows that it also satisfies State Independence. Hence, by Theorem 13.3 in Fishburn (1970), $\succ_{\sigma(F)}$ admits an EU representation (v,p). Furthermore, since $\succ_{\sigma(F)}$ and $\succ$ agree on X, v is cardinally equivalent to the utility u in Lemma 9, so one can take $v = u$.

3: let $x, y \in X$ be such that $\mathrm{E}_p u \circ f > u(x) > u(y) > \mathrm{E}_p u \circ g$. By part 2, $f \succ_{\sigma(F)} x \succ_{\sigma(F)} y \succ_{\sigma(F)} g$, and furthermore $x \succ y$ because u also represents $\succ$ on X. Since $F \setminus \sigma(F)$ is negligible given F by Corollary 6, by Remark 4 $x \succ y$ implies that $x\sigma(F)f \succ_F y\sigma(F)g$. Furthermore, by he definition of conditional preferences, $f \succ_{\sigma(F)} x$ iff $f \succ x\sigma(F)f$, i.e., iff $f \succ_F x\sigma(F)f$. Similarly, $y \succ_{\sigma(F)} g$ iff $y\sigma(F)g \succ_F g$. Thus, by Transitivity (Axiom 1 and Observation 4), $f \succ_F g$.

4: fix $N \in \Sigma$. Suppose that $p(N) = 0$. Consider $f,g \in \mathscr{A}$. Then $f(\omega) \succ g(\omega)$ for $\omega \notin N$ implies $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g$, so $f \succ_F g$ by part 3 of this Lemma. Hence N is negligible for F.

Conversely, suppose $p(N) > 0$. Pick $x, y \in X$ so $x \succ y$, and consider $f_\epsilon = \epsilon x + (1-\epsilon)y$ and $g = xNy$, where $\epsilon \in (0,1)$. For $\omega \notin N$, $f_\epsilon(\omega) = \epsilon x + (1-\epsilon)y \succ y = g(\omega)$. However, $\mathrm{E}_p\, u \circ f_\epsilon = \epsilon u(x) + (1-\epsilon)u(y)$ and $\mathrm{E}_p\, u \circ g = p(N)u(x) + [1-p(N)]u(y)$; for $\epsilon < p(N)$, $g \succ_F f_\epsilon$ by part 3 of this Lemma. Thus, N is not negligible given F. ■

Lemma 16 *Assume Axioms 1–8. For every n-sequence $F_1, \dots, F_L$, there exists a unique $p \in \Delta(\Sigma)$, with $p(\cup_{\ell=1}^L F_\ell) = 1$, such that*

1. *for every $f, g \in \mathscr{A}$, $\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g$ implies $f \succ_{\cup_{\ell=1}^L F_\ell} g$, whereu $: X \in \mathbb{R}$ is the function that represents $\succ$ on X per Lemma 9.*

2. *an event $N \in \Sigma$ is negligible given $\cup_{\ell=1}^L F_\ell$ if and only if $p(N) = 0$.*

3. *$p(\sigma(\cup_{\ell=1}^L F_\ell)) = 1$; moreover, for every $f, g \in \mathscr{A}$, $\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g$ implies $f \succ_{\sigma(\cup_{\ell=1}^L F_\ell)} g$.*

4. *$\succ_{\cup_{\ell=1}^L F_\ell}$ and $\succ_{\sigma(\cup_{\ell=1}^L F_\ell)}$ are non-degenerate.*

Proof: By Remark 5, if a $p \in \Delta(\Sigma)$ with the property in part 1 exists, it is unique.

1: for every $\ell = 1, \dots, L \in \mathscr{F}$, let $q_\ell \in \Delta(\Sigma)$ be the probability delivered by Lemma 15.

The statement follows from the following, slightly stronger claim: for every $K = 1, \dots, L$, there is $p^K \in \Delta(\Sigma)$ such that $p^K(\cup_{\ell=1}^K E_\ell) = 1$, $p^K(F_K) > 0$, $q_K = p^K(\cdot|F_K)$, and $\mathrm{E}_{p^K}\, u \circ f > \mathrm{E}_{p^K}\, u \circ g$ implies $f \succ_{\cup_{\ell=1}^K F_\ell} g$.

For $K = 1$, the claim follows from Lemma 15 parts 2 and 3. Thus, assume the claim holds for some $K \in \{1, \dots, L-1\}$, and let $p^K \in \Delta(\Sigma)$ be the corresponding probability. Since F_{K+1} is not negligible given F_K, Lemma 15 part 4 implies that $q_K(F_{K+1}) > 0$. The inductive hypothesis implies that $p^K(F_K) > 0$ and $p^K(F_{K+1} \cap F_K) = q_K(F_{K+1}) \cdot p^K(F_K) > 0$, so $p^K(F_{K+1}) > 0$.

If also $q_{F_{K+1}}(\cup_{\ell=1}^K F_\ell) > 0$, then Lemma 13 yields a new probability $p^{K+1} \in \Delta(\Sigma)$ with $p^{K+1}(\cup_{\ell=1}^{K+1} F_\ell) = 1$, such that $\mathrm{E}_{p^{K+1}}\, u \circ f > \mathrm{E}_{p^{K+1}}\, u \circ g$ implies $f \succ_{\cup_{\ell=1}^{K+1} F_\ell} g$. Furthermore, since p^{K+1} is the probability delivered by Lemma 12, in particular $p^{K+1}(F_{K+1}) > 0$ and $p^{K+1}(\cdot|F_{K+1}) = q_{K+1}$.

If instead $q_{F_{K+1}}(\cup_{\ell=1}^{K} F_\ell) = 0$, by Lemma 14 $\mathrm{E}_{q_{K+1}} u \circ f > \mathrm{E}_{q_{K+1}} u \circ g$ implies $f \succ_{\cup_{\ell=1}^{K+1} F_\ell} g$. Thus, one can take $p^{K+1} = q_{F_{K+1}}$. In this case, trivially $p^{K+1}(F_{K+1}) > 0$ and $p^{K+1}(\cdot|F_{K+1}) = q_{K+1}$.

To streamline notation, let $F = \cup_{\ell=1}^{L} F_\ell$ in the remainder of the proof.

2: fix $N \in \Sigma$. Suppose that $p(N) = 0$. Consider $f, g \in \mathscr{A}$. Then $f(\omega) \succ g(\omega)$ for $\omega \notin N$ implies $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g$, so $f \succ_F g$ by part 1 of this Lemma. Hence N is negligible for F.

Conversely, suppose $p(N) > 0$. Pick $x, y \in X$ so $x \succ y$, and consider $f_\epsilon = \epsilon x + (1-\epsilon) y$ and $g = xNy$, where $\epsilon \in (0,1)$. For $\omega \notin N$, $f_\epsilon(\omega) = \epsilon x + (1-\epsilon) y \succ y = g(\omega)$. However, $\mathrm{E}_p u \circ f_\epsilon = \epsilon u(x) + (1-\epsilon) u(y)$ and $\mathrm{E}_p u \circ g = p(N) u(x) + [1 - p(N)] u(y)$; for $\epsilon < p(N)$, $g \succ_F f_\epsilon$ by part 1 of this Lemma. Thus, N is not negligible given F.

3: suppose that $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g$. Since $F \setminus \sigma(F)$ is negligible given F by Corollary 4, $p(F \setminus \sigma(F)) = 0$ by part 2. Thus, $p(\sigma(F)) = p(F) = 1$. Hence, $\mathrm{E}_p u \circ f\sigma(F)z = \mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g = \mathrm{E}_p u \circ g\sigma(F)z$ for any $z \in X$. By part 1, $f\sigma(F)z \succ_F g\sigma(F)z$. Taking z as the act on the complement of F in Definition 5, $f\sigma(F)z \succ g\sigma(F)z$. Thus, again taking z as the act on the complement of F in Definition 5, $f \succ_{\sigma(F)} g$.

4: this follows from parts 1 and 3, because u is non-constant. ■

Next, we establish Eq. (3), the chain rule of conditioning.

Lemma 17 *Assume Axioms 1–8. Let $F_1, \ldots, F_L$ and $G_1, \ldots, G_M$ be n-sequences; denote by p and, respectively, q the probabilities whose existence is asserted by Lemma 16. For every $E \in \Sigma$, if $E \subseteq \cup_{m=1}^{M} G_m \subseteq \cup_{\ell=1}^{L} F_\ell$, then $p(E) = q(E) \cdot p(\cup_{m=1}^{M} G_m)$.*

Proof: Let $F = \cup_{\ell=1}^{L} F_\ell$ and $G = \cup_{m=1}^{M} G_m$, so $E \subseteq G \subseteq F$. If $p(G) = 0$, then also $p(E) = 0$ and there is nothing to show, so assume $p(G) > 0$.

Preliminary claim: for any event $D \subseteq G$, $q(D) = 0$ implies $p(D) = 0$. Proof: assume $q(D) = 0$. Then, by Lemma 16 part 2, D is negligible given G, and $D \subseteq G \subseteq F$; thus, by Lemma 11 part 2, D is also negligible given F. Then, Lemma 16 part 2 implies that $p(D) = 0$.

In particular, $q(E)=0$ implies $p(E)=0$. Moreover, since $G \setminus \sigma(G)$ is negligible given G by Corollary 6, by Lemma 16 part 2 $q(G \setminus \sigma(G))=0$, and thus also $p(G \setminus \sigma(G))=0$, hence $p(G)=p(\sigma(G))$.

Next, I claim that it is enough to consider $E \subseteq \sigma(G)$. Consider an arbitrary $E \in \Sigma$. Since, by Lemma 16 part 3, $q(\sigma(G))=1$, it must be the case that $q(E)=q(E \cap \sigma(G))$. Therefore, if $q(E \cap \sigma(G))p(G)=p(E \cap \sigma(G)$, then also $q(E)p(G)=q(E \cap \sigma(G))p(G)=p(E \cap \sigma(G))=p(E \setminus \sigma(G))+p(E \cap \sigma(G))=p(E)$, because, by the Preliminary Claim, $q(E \setminus \sigma(G))=0$ implies $p(E \setminus \sigma(G))=0$. Thus, henceforth, assume $E \subseteq \sigma(G)$.

Fix x,y with $x \succ y$. Let $\alpha < q(E)$, so that $u(x)q(E)+u(y)[1-q(E)] > \alpha u(x)+(1-\alpha)y$. Then, by Lemma 16 part 3, $xEy \succ_{\sigma(G)} \alpha x+(1-\alpha)y$. By Definition 5, taking y as the act on the complement of $\sigma(G)$, $(xEy)\sigma(G)y \succ [\alpha x+(1-\alpha)y]\sigma(G)y$. Applying Definition 5 again, with act y on the complement of $F \supseteq G \supseteq \sigma(G) \supseteq E$, $(xEy)\sigma(G)y \succ_F [\alpha x+(1-\alpha)y]\sigma(G)y$. Then Irreflexivity (Axiom 1 and Observation 4) and Lemma 16 part 1 imply that

$$p(E)u(x)+[1-p(E)]u(y) \geq p(\sigma(G))[\alpha u(x)+(1-\alpha)u(y)]+[1-p(\sigma(G))]u(y).$$

Rewrite:

$$p(E)[u(x)-u(y)]+u(y) \geq \alpha p(\sigma(G))[u(x)-u(y)]+u(y).$$

Since $u(x) > u(y)$, this holds if and only if $p(E) \geq \alpha p(\sigma(G))$. Since $p(\sigma(G))=p(G)>0$, conclude that $\alpha < q(E)$ implies $\frac{p(E)}{p(G)} \geq \alpha$. Similarly, $\alpha > q(E)$ implies $\frac{p(E)}{p(G)} \leq \alpha$. Therefore, $\frac{p(E)}{p(G)}=q(E)$. ■

The following immediate Corollary is key in the construction of a CPS.

Corollary 7 *If* $\cup_{\ell=1}^{L} F_\ell = \cup_{m=1}^{M} G_m$, *then* $p=q$.

Construction of the CCPS μ and plausibility relation $\geq^\mu$: let $\mathscr{F}^*=\{\cup_{\ell=1}^{L} F_\ell : F_1,\ldots,F_L$ is an n-sequence$\}$. Note that $\mathscr{F} \subseteq \mathscr{F}^*$. Define a function $\rho^*: \Sigma \times \mathscr{F}^* \to [0,1]$ by letting $\rho^*(E|\cup_{\ell=1}^{L} F_\ell)=$

$p(E)$ for each n-sequence $F_1,\ldots,F_L$, where p is the probability associated with $F_1,\ldots,F_L$ as per Lemma 16. By Corollary 7, this definition is well-posed (conditional probability depends only on the conditioning event, not on the specific n-sequence used to define it). Let μ denote the restriction of ρ^* to conditioning events in $\mathscr{F}$. Then, let $\geq^\mu \in \mathscr{F}\times\mathscr{F}$ be the plausibility relation induced by μ according to Def. 2.

Lemma 18 *Assume Axioms 1–8.*

1. $\rho^* \in \tilde{\Delta}(\Sigma,\mathscr{F}^*)$;
2. $N\in\Sigma$ *is negligible given* $F\in\mathscr{F}$ *iff* $\mu(N|F)=0$*. Hence* $F_1,\ldots,F_L\in\mathscr{F}$ *is an n-sequence iff it is a* μ*-sequence, a full sequence is a full* μ*-sequence, and Finally* $\mathscr{F}^*=\mathscr{F}_\mu$.
3. $\mu\in\Delta(\Sigma,\mathscr{F})$ *and, for every* $G\in\mathscr{F}$, $P_\mu(G)=\rho^*(\cdot|B_\mu(G))$;
4. $\rho^*=\rho$*, where* ρ *is the CPS defined from* $P_\mu(\cdot)$ *in Eq. (6).*

Furthermore, consider a μ*-sequence* $F_1,\ldots,F_L\in\mathscr{F}$*. Let* $u: X\to\mathbb{R}$ *be the function that represents* $\succ$ *on* X *per Lemma 9. Then*

5. $\forall f,g\in\mathscr{A}$, $\mathrm{E}_{\rho^*(\cdot|\cup_\ell F_\ell)}u\circ f > \mathrm{E}_{\rho^*(\cdot|\cup_\ell F_\ell)}u\circ g$ *implies* $f\succ_{\cup_{\ell=1}^L F_\ell} g$;
6. $N\in\Sigma$ *is negligible given* $\cup_{\ell=1}^L F_\ell$ *iff* $\rho^*(N|\cup_\ell F_\ell)=0$*, hence iff* $P_\mu(F_L)(N\cap[\cup_\ell F_\ell])=0$;
7. $\rho^*(\sigma(\cup_{\ell=1}^L F_\ell)|\cup_\ell F_\ell)=1$*, and* $\forall f,g\in\mathscr{A}$, $\mathrm{E}_{\rho^*(\cdot|\cup_\ell F_\ell)}u\circ f > \mathrm{E}_{\rho^*(\cdot|\cup_\ell F_\ell)}u\circ g$ *implies* $f\succ_{\sigma(\cup_{\ell=1}^L F_\ell)} g$.
8. $\sigma(\cup_\ell F_\ell)=\sigma_\mu(\cup_\ell F_\ell)$.

Proof: (1): by the definition of ρ^* and Lemma 16, for every $F\in\mathscr{F}^*$, $\rho^*(F|F)=1$. By Lemma 17, ρ^* satisfies Eq. (3): if $E\in\Sigma$ and $F,G\in\mathscr{F}^*$ are such that $E\subseteq F\subseteq G$, then $\rho^*(E|G)=\rho^*(E|F)\rho^*(F|G)$. Thus, $\rho^*\in\Delta(\Sigma,\mathscr{F}^*)$.

(2): consider the trivial n-sequence $F_1\equiv F$, where $F\in\mathscr{F}$. From the definition of $\mu(\cdot|F)=\rho^*(\cdot|F)$ and Lemma 16 part 2, $N\in\Sigma$ is negligible given F iff $\mu(N|F)=0$. The other statements follow immediately.

(3): for $F \in \mathscr{F}$, $\mu(F|F) = \rho^*(F|F) = 1$. Furthermore, consider $F_1, \dots, F_L \in \mathscr{F}$ and $E \subseteq F_1 \cap F_L$. First, consider the case in which $F_1, \dots, F_L$ is not a μ-sequence, hence not an n-sequence. Then there is $\ell_0 \in \{1, \dots, L-1\}$ such that $\mu(F_{\ell_0+1}|F_{\ell_0}) = 0$, so $\prod_{\ell=1}^{L-1} \mu(F_{\ell+1}|F_\ell) = 0$. If also $\prod_{\ell=1}^{L-1} \mu(F_\ell|F_{\ell+1}) = 0$, then Eq. 2 holds. Thus, suppose instead that $\prod_{\ell=1}^{L-1} \mu(F_\ell|F_{\ell+1}) > 0$, so $F_L, \dots, F_1$ is a μ-sequence and hence an n-sequence. Then $F \equiv \cup_n F_n \in \mathscr{F}^* = \mathscr{F}_\mu$. Since $1 = \rho^*(F|F) \le \sum_\ell \rho^*(F_\ell|F)$, there is n such that $\rho^*(F_\ell|F) > 0$. For any such ℓ, if $\ell > 1$, then $\rho^*(F_{\ell-1}|F) \ge \rho^*(F_{\ell-1} \cap F_\ell|F) = \rho^*(F_{\ell-1} \cap F_\ell|F_\ell) \cdot \rho^*(F_\ell|F) = \mu(F_{\ell-1}|F_\ell) \cdot \rho^*(F_\ell|F) > 0$, because by assumption $\mu(F_{\ell-1}|F_\ell) > 0$. Therefore, there is $m \in \{1, \dots, L\}$ such that $\rho^*(F_\ell|F) > 0$ for all $\ell = 1, \dots, m$. In particular $\rho^*(F_1|F) > 0$. By contradiction, suppose that $\mu(E|F_1) > 0$. Then by the chain rule $\rho^*(E|F) > 0$ as well, and since $E \subseteq F_1 \cap F_L$, $\rho^*(F_L|F) > 0$. Thus, $m = L$. In particular $\rho^*(F_{\ell_0+1}|F) > 0$, and since by assumption $\mu(F_{\ell_0}|F_{\ell_0+1}) > 0$, the chain rule implies that $\rho^*(F_{\ell_0+1} \cap F_{\ell_0}|F) = \mu(F_{\ell_0+1} \cap F_{\ell_0}|F_{\ell_0+1})\rho^*(F_{\ell_0+1}|F) > 0$. But then, again by the chain rule, $\mu(F_{\ell_0+1}|F_{\ell_0}) = \frac{\rho^*(F_{\ell_0+1} \cap F_{\ell_0}|F)}{\rho^*(F_{\ell_0+1}|F)} > 0$, contradiction. Therefore, $\mu(E|F_1) = 0$, and Eq. (2) holds again.

Finally, suppose that $F_1, \dots, F_L$ is a μ-sequence. If $F_L, \dots, F_1$ is not a μ-sequence, so $\prod_{\ell=1}^{L-1} \mu(F_\ell|F_{\ell+1}) = 0$, then a symmetric argument to the one just given shows that $\mu(E|F_L) = 0$ and so Eq. (2) holds. If instead $F_L, \dots, F_1$ is also a μ-sequence, then $\rho^*(F_\ell|F) > 0$ for all ℓ. The above argument shows that, since $F_L, \dots, F_1$ is a μ-sequence, there is m such that $\rho^*(F_\ell|F) > 0$ for all $\ell = 1, \dots, m$. Furthermore, suppose $\rho^*(F_\ell|F) > 0$ for some $\ell < L$. Since $F_1, \dots, F_L$ is a μ-sequence, $\mu(F_{\ell+1} \cap F_\ell|F_\ell) > 0$, and so by the chain rule $\rho^*(F_{\ell+1} \cap F_\ell|F) = \mu(F_{\ell+1} \cap F_\ell|F_\ell)\rho^*(F_\ell|F) > 0$, and so $\rho^*(F_{\ell+1}|F) > 0$ as well. Thus, $m = L$. Then, the measure $P \equiv \rho^*(\cdot|F)$ satisfies $P(F_\ell) > 0$ and $\mu(\cdot|F_\ell) = P(\cdot|F_\ell)$ for all ℓ, so the argument in Section 3.1 shows that Eq. (2) holds.

Finally, fix $G \in \mathscr{F}$. Then, by Remark 2 part 5 there is a μ-sequence $F_L, \dots, F_1$ (the numbering is chosen so as to match the arguments given above) such that $\{F_1, \dots, F_L\}$ is the equivalence class of $\ge^\mu$ containing G, such that $F_1 = F_L = F$. Then $B_\mu(G) = \cup_\ell F_\ell$. As shown above, if $\rho^*(F_\ell|B_\mu(G)) > 0$ and $\ell > 1$, then $\rho^*(F_{\ell-1}|B_\mu(G)) > 0$ as well. Moreover, $\rho^*(F_1|B_\mu(G)) > 0$. But since $F_1 = F_L = G$, $\rho^*(F_L|B_\mu(G)) > 0$, and so $\rho^*(F_\ell|B_\mu(G)) > 0$ for all ℓ. Since ρ^* is a CPS, $\mu(\cdot|F_\ell) = \frac{\rho^*(\cdot|B_\mu(G))}{\rho^*(F_\ell|B_\mu(G))}$ for all ℓ. Thus, $\rho^*(\cdot|B_\mu(G))$ satisfies all the properties in Proposition 1; but since there is a unique such measure, denoted $P_\mu(G)$ in Definition 3, it must be that $\rho^*(\cdot|B_\mu(G)) = P_\mu(G)$.

(4): since μ is a CCPS, $P_\mu(\cdot)$ satisfies the properties in Proposition 1 for all $F \in \mathscr{F}$, and one can define a CPS $\rho \in \Delta(\Sigma, \mathscr{F}_\mu)$ from $P_\mu(\cdot)$ via Eq. (6).

Now fix a μ-sequence $F_1, \dots, F_L$. Lemma 1 yields an m such that $F_\ell =^\mu F_L$ and $P_\mu(F_L)(F_\ell) > 0$ iff $\ell \geq m$. Hence $\cup_{\ell=m}^L F_\ell \subseteq B_\mu(F_L)$, and so, by Eq. (6) and the fact that $P_\mu(G) = \rho^*(\cdot|B_\mu(G))$ for every $G \in \mathscr{F}$,

$$\rho(E|\cup_\ell F_\ell) = \frac{P_\mu(F_L)(E \cap [\cup_{\ell=1}^L F_\ell])}{P_\mu(F_L)(\cup_{\ell=1}^L F_\ell)} = \frac{P_\mu(F_L)(E \cap [\cup_{\ell=m}^L F_\ell])}{P_\mu(F_L)(\cup_{\ell=m}^L F_\ell)} = \frac{\rho^*(E \cap [\cup_{\ell=m}^L F_\ell]|B_\mu(F_L))}{\rho^*(\cup_{\ell=m}^L F_\ell|B_\mu(F_L))}.$$

By Remark 2 part 5, there is a μ-sequence $F_{L+1}, \dots, F_{L+M}$ such that $\{F_{L+1}, \dots, F_{L+M}\}$ is the $\geq^\mu$-equivalence class of F_L, and $F_{L+1} = F_{L+M} = F_L$. Furthermore, by Remark 2 part 2, $F_1, \dots, F_{L+M}$ is also a μ-sequence.

I claim that $\rho^*(B_\mu(F_L)|\cup_{\ell=1}^{L+M} F_\ell) = 1$. If not, then since $B_\mu(F_L) = \cup_{\ell=L+1}^{L+M} F_\ell = \cup_{\ell=m}^{L+M} F_\ell$ because $F_\ell =^\mu F_L$ for $\ell = m, \dots, L$, there must be $n_0 \in \{1, \dots, m-1\}$ such that $\rho^*(F_{n_0}|\cup_{\ell=1}^{L+M} F_\ell) > 0$. Let n be the maximum such index n_0. Since $F_1, \dots, F_{L+M}$ is a μ-sequence, $0 < \mu(F_{n+1}|F_n) = \mu(F_{n+1} \cap F_m|F_n) = \rho^*(F_{n+1} \cap F_n|F_n)$. But then, by the chain rule, $\rho^*(F_{n+1}|\cup_{\ell=1}^{L+M} F_\ell) > 0$, which contradicts the choice of n. This proves the claim.

Then, by the chain rule,

$$\frac{\rho^*(E \cap [\cup_{\ell=m}^L F_\ell]|B_\mu(F_L))}{\rho^*(\cup_{\ell=m}^L F_\ell|B_\mu(F_L))} = \frac{\rho^*(E \cap [\cup_{\ell=m}^L F_\ell]|B_\mu(F_L)) \cdot \rho^*(B_\mu(F_L)|\cup_{\ell=1}^{L+M} F_\ell)}{\rho^*(\cup_{\ell=m}^L F_\ell|B_\mu(F_L)) \cdot \rho^*(B_\mu(F_L)|\cup_{\ell=1}^{L+M} F_\ell)} = \frac{\rho^*(E \cap [\cup_{\ell=m}^L F_\ell]|\cup_{\ell=1}^{L+M} F_\ell)}{\rho^*(\cup_{\ell=m}^L F_\ell|\cup_{\ell=1}^{L+M} F_\ell)}.$$

Moreover, $\rho^*(B_\mu(F_L)|\cup_{\ell=1}^{L+M} F_\ell) = 1$ implies that, for $n < m$, $\rho^*(F_n \setminus B_\mu(F_L)|\cup_{\ell=1}^{L+M} F_\ell) = 0$, and $\rho^*(F_n \cap B_\mu(F_L)|\cup_{\ell=1}^{L+M} F_\ell) = \rho^*(F_n \cap B_\mu(F_L)|B_\mu(F_L)) \cdot \rho^*(B_\mu(F_L)|\cup_{\ell=1}^{L+M} F_\ell) = P_\mu(F_L)(F_n \cap B_\mu(F_L)) = 0$; thus, $\rho^*(F_n|\cup_{\ell=1}^{L+M} F_\ell) = 0$ for $n < m$, so

$$\frac{\rho^*(E \cap [\cup_{\ell=m}^L F_\ell]|\cup_{\ell=1}^{L+M} F_\ell)}{\rho^*(\cup_{\ell=m}^L F_\ell|\cup_{\ell=1}^{L+M} F_\ell)} = \frac{\rho^*(E \cap [\cup_{\ell=1}^L F_\ell]|\cup_{\ell=1}^{L+M} F_\ell)}{\rho^*(\cup_{\ell=1}^L F_\ell|\cup_{\ell=1}^{L+M} F_\ell)} = \rho^*(E|\cup_{\ell=1}^L F_\ell),$$

where the last equality follows from the chain rule.

(5, 6), (7): since $\rho^*(\cdot|\cup_\ell F_\ell)$ is defined as the probability measure delivered by Lemma 16, the claims follow, respectively, from parts 1, 2, and 3 of that Lemma. The last part of 6 follows from part 4 and Eq. (6).

(8): by part (6), $\sigma(\cup_\ell F_\ell) = \cup\{[s_{-i}] : P_\mu(F_L)([s_{-i}] \cap (\cup_\ell F_\ell)) > 0\} = \sigma_\mu(\cup_\ell F_\ell)$. ■

For the remainder of this Section, *I will not distinguish between n-sequences and μ-sequences, between full sequences and full μ-sequences, or between σ and σ_μ*. I will also invoke the definitions and results in Section A, as they only rely upon the properties of CCPSs.

Lemma 19 *Assume Axioms 1–8. Fix $F, G_1, \ldots, G_K \in \mathscr{F}$ such that $F >^\mu G_k$ for every $k = 1, \ldots, K$. Then $\cup_{k=1}^K G_k$ is negligible given $B_\mu(F) \cup [\cup_{k=1}^K G_k]$.*

The set $B_\mu(F) \cup [\cup_k G_k]$ is non-degenerate by Corollary 3, because it contains $F \in \mathscr{F}$.

Proof: For every k, there is a μ-sequence $G_1^k, \ldots, G_{M^k}^k$ such that $G_1^k = G_k$ and $G_{M^k}^k = F$. Let $F_1, \ldots, F_L$ be the $\geq$-equivalence class of $F = G_{M^k}^k$, arranged per Remark 2 part 5 so that $F_1 = F_L = F$. Then $G_1^k, \ldots, G_{M^k}^k, F_1, \ldots, F_L$ is also a μ-sequence by part 2 of that Remark. Let $G^k = \cup_{m=1}^{M^k} G_m^k$ and $E = B_\mu(F) \cup G^k$. By Lemma 3, there is $\bar{m}$ such that $G_m^k =^\mu F_L = F$ iff $m \geq \bar{m}$; moreover, for $m < \bar{m}$, $P_\mu(F)(G_m^k) = \rho^*(G_m^k|E) = 0$, where $\rho = \rho^*$ by Lemma 18 part 4. Since by assumption $F >^\mu G_k = G_1^k$, in particular $1 < \bar{m}$, and so $P_\mu(F)(G_k) = \rho^*(G_k|E) = 0$.

By part 6 of Lemma 18, G_k is negligible given E. Furthermore, by the same Lemma, $\rho^* = \rho$ satisfies Equation (6), so $\rho^*(B_\mu(F)|E) = \frac{P_\mu(F)(B_\mu(F) \cap E)}{P_\mu(F)(E)} = 1$, because $B_\mu(F) \subseteq E$. A fortiori $\rho^*(B_\mu(E) \cup G_k|E) = 1$. Therefore, for all $x, y, z \in X$ with $y \succ z$, $u(y)\rho^*(B_\mu(F) \cup G_k)|E) + u(z)[1 - \rho^*(B_\mu(F) \cup G_k|E)] = u(y) > u(z) = u(x)\rho^*(G_k|E) + u(z)[1 - \rho^*(G_k|E)]$. By part 5 of Lemma 18, $y(B_\mu(F) \cup G_k)z \succ_E xG_kz$. By the definition of conditional preferences, $y(B_\mu(F) \cup G_k)z \succ xG_kz$, and again by the definition of conditional preferences, $y \succ_{B_\mu(F) \cup G_k} xG_kz$. If not $x \succ z$, then Axiom 4 (Monotonicity) implies $y \succ_{B_\mu(F) \cup G_k} x$; otherwise, by Irreflexivity not $z \succ x$, and the same Axiom implies $y \succ_{B_\mu(F) \cup G_k} z$. In either case, $\succ_{B_\mu(F) \cup G_k}$ is non-degenerate, so by Lemma 10, G_k is negligible for $B_\mu(F) \cup G_k$.

To complete the proof, I show inductively that, for every $\bar{k} = 1, \ldots, K$, $\cup_{k=1}^{\bar{k}} G_k$ is negligible given $B_\mu(F) \cup [\cup_{k=1}^{\bar{k}} G_k]$. For $\bar{k} = 1$, the claim reduces to the assertion that G_1 is negligible given $B_\mu(F) \cup G_1$, which was established above. Inductively, assume the claim is true for

$\bar{k} \in \{1, \ldots, K-1\}$, and consider $\bar{k}+1$. By the inductive hypothesis, $\cup_{k=1}^{\bar{k}} G_k$ is negligible given $F \cup [\cup_{k=1}^{\bar{k}} G_k]$; moreover, it was shown above that $F \cup G_{\bar{k}+1}$ is negligible given $F \cup G_{\bar{k}+1}$. Hence, by Lemma 11 part 2, both $\cup_{k=1}^{\bar{k}} G_k$ and $G_{\bar{k}+1}$ are negligible given $F \cup [\cup_{k=1}^{\bar{k}+1} G_k]$. But then, part 3 in the same Lemma implies that $\cup_{k=1}^{\bar{k}+1} G_k$ is negligible given $F \cup [\cup_{k=1}^{\bar{k}+1} G_k$, as required. ■

It is now possible to **prove sufficiency in Theorems 3 and 2**. Again, I use Observation 1.

Assume $f \succ^{u,\mu} g$.

Let $\mathscr{F}^+ = \{F \in \mathscr{F} : \mathrm{E}_{P_\mu(F)(\cdot)}[u \circ f - u \circ g] > 0\}$. Let $\{F_1, \ldots, F_R\}$ be such that, for every $F \in \mathscr{F}^+$, $F =^\mu F_r$ for some r, and $F_r \neq^\mu F_\ell$ for $r \neq \ell$. Let $\Omega_0 = \Omega \setminus \cup_r \sigma(B_\mu(F_r))$ and, inductively, for $r \geq 1$,

$$\Omega_r^- = \left\{\omega \in \Omega_{r-1} \,:\, f(\omega) \prec g(\omega),\, \exists G_r^\omega \in \mathscr{F} : \omega \in G_r^\omega, F_r \geq^\mu G_r^\omega \right\}, \quad \Omega_{r+1} = \Omega_r \setminus \Omega_r^-.$$

By definition, if $f(\omega) \prec g(\omega)$ for some ω, there are $F, G \in \mathscr{F}$ such that $F \geq^\mu G$ and $\mathrm{E}_{P_\mu(F)(\cdot)}[u \circ f - u \circ g] > 0$. Let r be such that $F =^\mu F_r$. Then there are three possibilities: (i) $\omega \in \sigma(B_\mu(F_\ell))$ for some $\ell \in \{1, \ldots, R\}$; (ii) $\omega \in \Omega_r^-$, or (iii) $\omega \in \Omega_\ell^-$ for some $\ell < r$. Thus, for every ω, if $f(\omega) \prec g(\omega)$, then $\omega \in \cup_r [\sigma(B_\mu(F_r)) \cup \Omega_r^-]$.

I claim that, for $r \neq \ell$, $[\sigma(B_\mu(F_r)) \cup \Omega_r^-] \cap [\sigma(B_\mu(F_\ell)) \cup \Omega_\ell^-] = \emptyset$. By Lemma 5, $\sigma(B_\mu(F_r)) \cap \sigma(B_\mu(F_\ell)) = \emptyset$. By construction, $\Omega_r^- \cap \Omega_\ell^- = \emptyset$. Furthermore, $\Omega_r^- \cup \Omega_\ell^- \subseteq \Omega_0 = \Omega \setminus \cup_{r'} \sigma(B_\mu(F_{r'})) \subseteq \Omega \setminus [\sigma(B_\mu(F_r)) \cup \sigma(B_\mu(F_\ell))]$. This proves the claim.

For every $\omega \in \Omega_r^-$, either $G_r^\omega =^\mu F_r$, or $F_r >^\mu G_r$. In either case, $\omega \notin \sigma(B_\mu(F))$, because $\Omega_r^- \subseteq \Omega_0 = \Omega \setminus \cup_{\ell=1}^R \sigma(B_\mu(F_\ell))$. Define the sets

$$\Omega_r^> = \{\omega \in \Omega_r^- : F_r >^\mu G_r^\omega\} \quad \text{and} \quad \Omega_r^= = \{\omega \in \Omega_r^- : F_r =^\mu G_r^\omega\},$$

so $\Omega_r^- = \sigma(F) \cup \Omega_r^> \cup \Omega_r^=$.

If $\omega \in \Omega_r^=$, then $\omega \in G_r^\omega \subseteq B_\mu(F)$ by definition; however, $\Omega_r^= \subseteq \Omega_r^-$, so $\omega \notin \sigma(B_\mu(F_r))$. Therefore, $\Omega_r^= \subseteq B_\mu(F_r) \setminus \sigma(B_\mu(F_r))$, and so $\Omega_r^=$ is negligible given $B_\mu(F_r)$ by Corollary 5 ($\succ_{B_\mu(F_r)}$ is non-degenerate by part 4 of Lemma 16) and Lemma 11 part 1. I claim that $\Omega_r^>$ is negligible given $B_\mu(F_r) \cup \Omega_r^>$, where the latter set is non-degenerate by Corollary 3, because it contains $F_r \in \mathscr{F}$.

To show this, note first that, since $\mathscr{F}$ is finite, there are finitely many $\omega(1),\ldots,\omega(K)\in\Omega_r^>$ such that $\Omega_r^>\subseteq\cup_k G_r^{\omega(k)}$. By definition, $F_r >^\mu G_r^{\omega(k)}$ for each k. Lemma 19 then implies that $\cup_k G_r^{\omega(k)}$ is negligible given $B_\mu(F_r)\cup[\cup_k G_r^{\omega(k)}]$; the latter set is non-degenerate by Corollary 3, because it contains $F_r\in\mathscr{F}$. Lemma 11 part 1 implies that $\Omega_r^>$ is negligible given $B_\mu(F_r)\cup[\cup_k G_r^{\omega(k)}]$. By the same result, $\{B_\mu(F_r)\cup[\cup_k G_r^{\omega(k)}]\}\setminus\{B_\mu(F_r)\cup\Omega_r^>\}\subseteq\cup_k G_r^{\omega(k)}$ is negligible given $B_\mu(F_r)\cup[\cup_k G_r^{\omega(k)}]$; thus, finally, by part 4 in the same Lemma, $\Omega_r^>$ is negligible given $B_\mu(F_r)\cup\Omega_r^>$.

It follows that, by Lemma 11 part 2, both $\Omega_r^=$ and $\Omega_r^>$ are negligible given $B_\mu(F_r)\cup\Omega_r^>$; hence, by part 3 of the same Lemma, $\Omega_r^=\cup\Omega_r^>$ is negligible given $B_\mu(F_r)\cup\Omega_r^>=\sigma(B_\mu(F_r))$. Now note that

$$B_\mu(F_r)\cup\Omega_r^>=\sigma(B_\mu(F_r))\cup[B_\mu(F_r)\setminus\sigma(B_\mu(F_r))]\cup\Omega_r^>\supseteq\sigma(B_\mu(F_r))\cup\Omega_r^=\cup\Omega_r^>,$$

because $\Omega_r^=\subseteq B_\mu(F_r)\setminus\sigma(B_\mu(F_r))$. Therefore, $D_r\equiv[B_\mu(F_r)\cup\Omega_r^>]\setminus[\sigma(B_\mu(F_r))\cup\Omega_r^=\cup\Omega_r^>]\subseteq B_\mu(F_r)\setminus\sigma(B_\mu(F_r))$. By Corollary 6, $B_\mu(F_r)\setminus\sigma(B_\mu(F_r))$ is negligible given $B_\mu(F_r)$, and hence given $B_\mu(F_r)\cup\Omega_r^>$ by Lemma 11 part 2. Therefore, D_r is negligible given $B_\mu(F_r)\cup\Omega_r^>$ by part 1 of the same Lemma. Finally, $\Omega_r^=\cup\Omega_r^>=\Omega_r^-$ is negligible given $\sigma(B_\mu(F_r))\cup\Omega_r^=\cup\Omega_r^>=\sigma(B_\mu(F_r))\cup\Omega_r^-$ by Lemma 11 part 4.

Now let $x,y\in X$ be such that $\mathrm{E}_{P_\mu(F)(\cdot)}u\circ f>u(x)>u(y)>\mathrm{E}_{P_\mu(F)(\cdot)}u\circ g$. Then $f\succ_{\sigma(B_\mu(F))}x\succ_{\sigma(B_\mu(F))}y\succ_{\sigma(B_\mu(F))}g$ by Lemma 18 part 7 (recall that $\rho^*(\cdot|B_\mu(F))=P_\mu(F)(\cdot)$ by construction), and moreover $x\succ y$ because u is the utility function that represents $\succ$ on X by Lemma 9. Since Ω_r^- is negligible given $\sigma(B_\mu(F))\cup\Omega_r^-$, and these events are disjoint, $x\sigma(B_\mu(F))f\succ_{\sigma(B_\mu(F))\cup\Omega_r^-}y\sigma(f)g$. Moreover, $f\succ_{\sigma(B_\mu(F))}x$ iff $f\succ x\sigma(B_\mu(F))f$ iff $f\succ_{\sigma(B_\mu(F))\cup\Omega_r^-}x\sigma(B_\mu(F))f$, and similarly $y\succ_{\sigma(B_\mu(F))}g$ iff $y\sigma(B_\mu(F))g\succ_{\sigma(B_\mu(F))\cup\Omega_r^-}g$. Hence, by Transitivity, $f\succ_{\sigma(B_\mu(F))\cup\Omega_r^-}g$.

Conclude that, for every r, $f\succ_{\sigma(B_\mu(F_r))\cup\Omega_r^-}g$.

The sets $E_r\equiv\sigma(B_\mu(F_r))\cup\Omega_r^-$ are mutually disjoint. Apply the definition of conditional preferences repeatedly:

$$f \succ_{E_1} g \Rightarrow fE_1 g \succ g,$$

$$f \succ_{E_2} g \Rightarrow fE_2(fE_1 g) \succ gE_2(fE_1 g) \Rightarrow f(E_1 \cup E_2)g \succ fE_1 g,$$

$$\dots$$

$$f \succ_{E_r} g \Rightarrow fE_r[f(\cup_{\ell=1}^{r-1} E_\ell)g] \succ gE_r[f(\cup_{\ell=1}^{r-1} E_\ell)g] \Rightarrow f(\cup_{\ell=1}^{r} E_\ell)g \succ f(\cup_{\ell=1}^{r-1} E_\ell)g$$

$$\dots$$

$$f \succ_{E_R} g \Rightarrow fE_R[f(\cup_{\ell=1}^{R-1} E_\ell)g] \succ gE_R[f(\cup_{\ell=1}^{R-1} E_\ell)g] \Rightarrow f(\cup_{\ell=1}^{R} E_\ell)g \succ f(\cup_{\ell=1}^{R-1} E_\ell)g$$

Therefore, by Transitivity (Axiom 1), $f(\cup_r E_r)g \succ g$.

Finally, recall that $f(\omega) \prec g(\omega)$ implies $\omega \in E_r$ for some r. Therefore, for all $\omega \notin \cup_r E_r$, not $g(\omega) \succ f(\omega)$. Then, by Generalized Monotonicity (Remark 3), $f \succ g$.

Conversely, assume that not $f \succ^{u,\mu}$; it will be shown that not $f \succ g$. First, recall that, by Lemma 18 part 2, a full sequence is a full μ-sequence.

Remark 6 *If $F_1, \dots, F_L \in \mathscr{F}$ is a full sequence, then $\mathrm{E}_{\rho^*(\cdot|\cup_\ell F_\ell)} u \circ f = \mathrm{E}_{P_\mu(F_L)(\cdot)} u \circ f$ for all $f \in \mathscr{A}$.*

Proof: Since ρ^* satisfies Eq. (6) by Lemma 18 part 4, $\mathrm{E}_{\rho^*(\cdot|\cup_\ell F_\ell)} u \circ f = \frac{\mathrm{E}_{P_\mu(F_L)(\cdot)} u \circ f}{P_\mu(F_\ell)(\cup_\ell F_\ell)}$. But since $F_1, \dots, F_L$ is a full sequence, it is a full μ-sequence, so by Lemma 4 $B_\mu(F_L) \subseteq \cup_\ell F_\ell$, so $P_\mu(F_\ell)(\cup_\ell F_\ell) = 1$. ■

To complete the proof, there are two cases. The first is that, for all ω, not $f(\omega) \prec g(\omega)$. Then it must be that $\mathrm{E}_{P_\mu(F)(\cdot)}[u \circ f - u \circ g] = 0$ for all $F \in \mathscr{F}$. By contradiction, suppose $f \succ g$. Taking $E = \emptyset$, Axiom 9 implies that there is a full sequence $F_1, \dots, F_L \in \mathscr{F}$ such that $f \succ_{\cup_\ell F_\ell} x \succ_{\cup_\ell F_\ell} y \succ_{\cup_\ell F_\ell} g$. By Lemma 18 part 5, $\mathrm{E}_{\rho^*(\cdot|\cup_\ell F_\ell)} u \circ f \geq u(x)$ and $u(y) \geq \mathrm{E}_{\rho^*(\cdot|\cup_\ell F_\ell)}$. By Remark 6, $\mathrm{E}_{P_\mu(F_L)(\cdot)} u \circ f \geq u(x)$ and $u(y) \geq \mathrm{E}_{P_\mu(F_L)(\cdot)}$. Finally, $x \succ_{\cup_\ell F_\ell} y$ implies that $\succ_{\cup_\ell F_\ell)}$ is non-degenerate (this also follows from Corollary 3), so Lemma 9 implies that $u(x) > u(y)$. But then $\mathrm{E}_{P_\mu(F_L)(\cdot)} u \circ f > \mathrm{E}_{P_\mu(F_L)(\cdot)}$, contradiction. Thus, not $f \succ g$.

In the second case, $u(f(\omega)) < u(g(\omega))$, i.e., $f(\omega) \prec g(\omega)$, for some ω, but there are no $F, G \in \mathscr{F}$ with $F \geq^{\mu} G$, $\omega \in G$, and $\mathrm{E}_{P_\mu(F)(\cdot)}[u \circ f - u \circ g] > 0$. Again, suppose $f \succ g$. Taking $E = \{\omega\}$, Axiom 9 implies that there is a full sequence $F_1, \ldots, F_L \in \mathscr{F}$ such that $\{\omega\} \cap F_1 \neq \emptyset$, i.e., $\omega \in F_1$, and $f \succ_{\cup_\ell F_\ell} x \succ_{\cup_\ell F_\ell} y \succ_{\cup_\ell F_\ell} g$. As in the preceding case, this implies $\mathrm{E}_{P_\mu(F)(\cdot)}[u \circ f - u \circ g] > 0$. Since $F_1, \ldots, F_L$ is a full sequence, it is an n-sequence and hence a μ-sequence; thus, $F_L \geq^{\mu} F_1$. Taking $F = F_L$ and $G = F_1$ yields a contradiction. Thus, not $f \succ g$

This completes the proof.

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