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VECTOR EXPECTED UTILITY AND ATTITUDES TOWARD VARIATION

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This paper proposes a model of decision under ambiguity deemed *vector expected utility*, or VEU. In this model, an uncertain prospect, or Savage act, is assessed according to (a) a *baseline expected-utility evaluation*, and (b) an *adjustment* that reflects the individual's perception of ambiguity and her attitudes toward it. The adjustment is itself a function of the act's exposure to distinct sources of ambiguity, as well as its variability. The key elements of the VEU model are a baseline probability and a collection of random variables, or *adjustment factors*, which represent acts exposed to distinct ambiguity sources and also reflect complementarities among ambiguous events. The adjustment to the baseline expected-utility evaluation of an act is a function of the covariance of its utility profile with each adjustment factor, which reflects exposure to the corresponding ambiguity source.

A behavioral characterization of the VEU model is provided. Furthermore, an updating rule for VEU preferences is proposed and characterized. The suggested updating rule facilitates the analysis of sophisticated dynamic choice with VEU preferences.

KEYWORDS: Ambiguity, attitudes toward variability, reference prior.

1. INTRODUCTION

THE ISSUE OF AMBIGUITY in decision-making has received considerable attention in recent years, both from a theoretical perspective and in applications to contract theory, information economics, finance, and macroeconomics. As Ellsberg (1961) first observed, individuals may find it difficult to assign probabilities to events when available information is scarce or unreliable. In these circumstances, agents may avoid taking actions whose ultimate outcomes depend crucially upon the realization of such ambiguous events and instead opt for safer alternatives. Several decision models have been developed to accommodate these patterns of behavior: these models represent ambiguity via multiple priors (Gilboa and Schmeidler (1989), Ghirardato, Maccheroni, and Marinacci (2004)), nonadditive beliefs (Schmeidler (1989)), second-order probabilities (Klibanoff, Marinacci, and Mukerji (2005), Nau (2006), Ergin and Gul (2009)), relative entropy (Hansen and Sargent (2001), Hansen, Sargent, and Tallarini (1999)), or variational methods (Maccheroni, Marinacci, and Rustichini (2006)).

This paper proposes a decision model that incorporates key insights from Ellsberg's original analysis, as well as from cognitive psychology and recent theoretical contributions on the behavioral implications of ambiguity. According

[1]This is a substantially revised version of Siniscalchi (2001). Many thanks to Stephen Morris and three anonymous referees, as well as to Eddie Dekel, Paolo Ghirardato, Faruk Gul, Lars Hansen, Peter Klibanoff, Alessandro Lizzeri, Fabio Maccheroni, Massimo Marinacci, and Josè Scheinkman. All errors are my own.

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to the proposed model, the individual evaluates uncertain prospects, or acts, by a process suggestive of anchoring and adjustment (Tversky and Kahneman (1974)). The anchor is the expected utility of the prospect under consideration, computed with respect to a baseline probability; the adjustment depends upon its exposure to distinct sources of ambiguity, as well as its variation away from the anchor at states that the individual deems ambiguous. Formally, an act f, mapping each state $\omega \in \Omega$ to a consequence $x \in X$, is evaluated via the functional

$$V(f) = \mathrm{E}_p[u \circ f] + A\big((\mathrm{E}_p[\zeta_i \cdot u \circ f])_{0 \le i < n}\big). \tag{1}$$

In Eq. (1), $u : X \to \mathbb{R}$ is a von Neumann–Morgenstern utility function; p is a *baseline probability* on Ω, and E_p is the corresponding expectation operator; $n \le \infty$ and, for $0 \le i < n$, ζ_i is a random variable, or *adjustment factor*, that satisfies $\mathrm{E}_p[\zeta_i] = 0$; and the function $A : \mathbb{R}^n \to \mathbb{R}$ satisfies $A(0) = 0$ and $A(-\phi) = A(\phi)$ for every vector $\phi \in \mathbb{R}^n$. I call the proposed model *vector expected utility*, or VEU. This paper provides a behavioral characterization of preferences that conform to the VEU model; it also illustrates how tractable specifications of VEU preferences can reflect a variety of attitudes toward ambiguity and also facilitate the analysis of dynamic choice.

The remainder of this Introduction elaborates upon key features of the proposed model.

Anchoring and Adjustment: Einhorn and Hogarth (Einhorn and Hogarth (1985, 1986), Hogarth and Einhorn (1990)) were the first to propose that evaluating prospects by means of a baseline prior, adjusted to account for ambiguity, was a plausible approach to decisions under ambiguity. The cited papers explore the implications of this strategy in a series of experiments, dealing primarily with choice among binary lotteries. Ellsberg's seminal paper also suggests that, when faced with an ambiguous choice situation, "by compounding various probability judgments of various degrees of reliability, [the individual] can eliminate certain probability distributions over states of nature as 'unreasonable,' assign weights to others and *arrive at a composite 'estimated' distribution*" (Ellsberg (1961, p. 661); italics added for emphasis). Other authors have emphasized reference priors: see Section 5.1.

Adjustment Factors ζ_i and Eventwise Complementarity: Decomposing the adjustment term in Eq. (1) into a suitable function $A(\cdot)$ and a collection $(\zeta_i)_{0 \le i < n}$ of adjustment factors provides a direct, explicit representation of *eventwise complementarity*—a key behavioral feature of ambiguous events highlighted in the analysis of Epstein and Zhang (2001). To illustrate this notion and provide a simple application of the decision model of Eq. (1), consider Ellsberg's three-color-urn experiment. A ball is to be drawn from an urn containing 30 red balls, and 60 blue and green balls; the proportion of blue vs. green balls is unknown. Denote by f_R, f_B, f_{RG}, f_{BG} the acts that yield \$10 if a red (resp. blue, red or green, blue or green) ball is drawn, and \$0 otherwise. As reported by Ellsberg, the modal preferences are $f_R \succ f_B$ and $f_{RG} \prec f_{BG}$. Epstein and Zhang suggested that "[t]he intuition for this reversal is the complementarity between G and B—there is imprecision regarding the likelihood of B, whereas $\{B, G\}$ has precise probability $\frac{2}{3}$" (Epstein and Zhang (2001, p. 271)). The proposed model enables a representation of the modal preferences in this example that closely matches this interpretation: let p be uniform on the state space $\Omega = \{R, G, B\}$, assume without loss of generality (w.l.o.g.) that u is linear, and let ζ_0 be the random variable given by

$$\zeta_0(R) = 0, \quad \zeta_0(B) = 1, \quad \zeta_0(G) = -1.$$

Finally, let $A(\phi) = -|\phi|$ for every $\phi \in \mathbb{R}$. Thus, in this example, $n = 1$: one-dimensional adjustment factors suffice. The interpretation of the adjustment factor ζ_0 is as follows: since $A(p(\{G\})\zeta_0(G)) = A(p(\{B\})\zeta_0(B))$, G and B are "equally ambiguous"; however, $\zeta_0(G) = -\zeta_0(B)$, that is, their ambiguities "cancel out." This algebraic cancellation corresponds to Epstein and Zhang's notion of complementarity. It is then easily verified that $V(f_R) = \frac{10}{3}$, $V(f_B) = 0$, $V(f_{RG}) = \frac{10}{3}$, and $V(f_{BG}) = \frac{20}{3}$, which is consistent with the preferences indicated above.[2]

Adjustment Factors ζ_i and Sources of Ambiguity: Each factor ζ_i encodes a particular pattern of complementarity and thus reflects a specific aspect of ambiguity. Different considerations lead to a similar intuition. Since $\mathrm{E}_p[\zeta_i] = 0$ for all i, Eq. (1) can be rewritten in the form

$$V(f) = \mathrm{E}_p[u \circ f] + A\big((\mathrm{Cov}_p(\zeta_i, u \circ f))_{0 \le i < n}\big), \tag{2}$$

where Cov_p denotes covariance with respect to the baseline probability p. This suggests the following interpretation: each adjustment factor ζ_i is a "model" of ambiguous utility profile, whose evaluation is affected by a distinct[3] *source of ambiguity*; the adjustment applied to the baseline evaluation of an act f depends upon the similarity (as measured by covariance) of its utility profile $u \circ f$ with each factor ζ_i, and hence upon its exposure to the corresponding source of ambiguity. It may be useful to draw a parallel with factor-pricing models in finance: for instance, in the capital-asset pricing model (cf. Cochrane (2001, Section 9.1)), the expected return on an asset is a function of the covariance of its returns with the returns on the wealth portfolio.[4]

The construction of the adjustment factors in the proof of the characterization theorem (Theorem 1) supports this interpretation: as illustrated in Section 4.1, $(\zeta_i)_{0 \le i < n}$ is an orthonormal basis for a subspace of "purely ambiguous"

[2]For instance, $V(f_{RG}) = 10 \cdot \frac{2}{3} - |0 \cdot 10 \cdot \frac{1}{3} + 1 \cdot 0 \cdot \frac{1}{3} + (-1) \cdot 10 \cdot \frac{1}{3}| = \frac{20}{3} - |-\frac{10}{3}| = \frac{10}{3}$.

[3]In a "sharp" VEU representation, the factors ζ_i are *orthonormal*; this emphasizes the interpretation as *distinct* (uncorrelated) sources of ambiguity. See Definitions 1 and 2 for details.

[4]I thank Adam Szeidl for suggesting this analogy and the term "factor" to indicate the random variables ζ_i.

acts, and the expectations $\mathrm{E}_p[\zeta_i \cdot u \circ f] = \mathrm{Cov}_p[\zeta_i, u \circ f]$ are the Fourier coefficients of the projection of $u \circ f$ onto this subspace.

Adjustments and Variability: As noted above, adjustments to the baseline expected utility (EU) evaluation of an act are also related to the *variability*, or dispersion, of its utility profile. This can be attractive, as many economic applications of ambiguity-sensitive decision models show that interesting patterns of behavior can arise when agents wish to reduce outcome or utility variability.[5] Indeed, Schmeidler (1989, p. 582) suggested that "ambiguity aversion" can be defined as a preference for "smoothing or averaging utility distributions"; see also Chateauneuf and Tallon (2002).

The VEU representation relates adjustments to utility variability via two complementary channels. One is immediate from Eq. (2): the covariance of ζ_i and $u \circ f$ clearly depends upon the standard deviation of $u \circ f$ with respect to the baseline prior p.

The second channel deserves further discussion. Call two acts f and $\bar{f}$ *complementary* if their utility profiles $u \circ f$ and $u \circ \bar{f}$ satisfy $u \circ \bar{f} = c - u \circ f$ for some real constant c: Definition 3 provides a simple behavioral characterization. Notice that the utility profiles of f and $\bar{f}$ have the same standard deviation; indeed, virtually all classical measures of variability or dispersion for random variables[6] consider $u \circ f$ and $u \circ \bar{f} = c - u \circ f$ to be just as dispersed, because such measures are invariant to translation and sign changes. To relate adjustments to utility variability, the VEU representation incorporates the same invariance property: *complementary acts receive the same adjustment*. This follows from the symmetry property of the adjustment functional A: for every vector ϕ, $A(\phi) = A(-\phi)$.[7] Behaviorally, this property corresponds to the main novel axiom in this paper, *complementary independence*.

Behavioral Identification of the Baseline Prior p: One additional consequence of this property, and indeed of the Complementary Independence axiom, deserves special emphasis. Symmetry implies that adjustment terms cancel out when comparing two complementary acts using the VEU representation in Eq. (1); thus, the ranking of complementary acts is effectively determined by their baseline EU evaluation. Conversely, preferences over complementary acts uniquely identify the baseline prior: there is a unique probability p and a cardinally unique utility function u such that, for all complementary acts f and $\bar{f}$, $f \succcurlyeq \bar{f}$ if and only if (iff) $\mathrm{E}_p[u \circ f] \geq \mathrm{E}_p[u \circ \bar{f}]$. Thus, baseline priors have a simple behavioral interpretation in the present setting: *they provide a representation of the individual's preferences over complementary acts.* This

[5]See, for example, Bose, Ozdenoren, and Pape (2006), Epstein and Schneider (2007), Ghirardato and Katz (2006), or Mukerji (1998).

[6]For instance, the mean absolute deviation, the range and (for continuous random variables) the interquantile range, Gini's mean difference (cf., e.g., Yitzhaki (1982)), or peakedness ordering Bickel and Lehmann (1976).

[7]Notice that if f and $\bar{f}$ are complementary, then $\mathrm{Cov}(\zeta_i, u \circ \bar{f}) = -\mathrm{Cov}(\zeta_i, u \circ f)$ for all ζ_i.

implies that, under complementary independence, the baseline prior is behaviorally identified independently of other elements of the VEU representation.

Flexibility and Dynamics: Finally, the functional representation in Eq. (1) is flexible enough to accommodate a broad range of attitudes toward ambiguity, while at the same time allowing for numerical and analytical tractability. The preferences in the three-color-urn example display ambiguity aversion as defined by Schmeidler (1989); correspondingly, the adjustment function A is nonpositive and concave. VEU preferences featuring a nonpositive and concave adjustment function A are variational (Maccheroni, Marinacci, and Rustichini (2006), Corollary 2 and Section 5.1), but VEU preferences allow for considerably more general ambiguity attitudes. For instance, as shown in Section 4.3, a nonpositive, but not necessarily concave adjustment function characterizes "comparative ambiguity aversion" in the sense of Ghirardato and Marinacci (2002); a parsimonious VEU representation with this property can, for instance, accommodate the interesting preference patterns highlighted by Machina (2009) (such patterns are inconsistent with decision models such as maxmin expected utility, variational preferences or smooth-ambiguity-averse preferences (cf. Baillon, L'Haridon, and Placido (2008)). Indeed, the VEU model can accommodate even more complex attitudes towards ambiguity—for instance, stake-dependent attitudes: the previous version of this paper (Siniscalchi (2007)) provides an example.

This paper also proposes a possible *updating rule* for VEU preferences and provides a behavioral characterization. In the covariance formulation of the VEU model in Eq. (2), the proposed rule amounts to replacing expectations and covariances E_p and Cov_p with their conditional counterparts $\mathrm{E}_p[\cdot|E]$ and $\mathrm{Cov}_p(\cdot,\cdot|E)$.[8] Section 4.4 provides a behavioral characterization of this updating rule; it also illustrates how this rule enables a *recursive* analysis of sophisticated choice in dynamic problems.

The paper is organized as follows. Section 2 is devoted to preliminaries. Section 3 presents the main characterization result. Section 4 analyzes the components of the VEU representation (Sections 4.1–4.3), and discusses updating and dynamic choice (Sections 4.4 and 4.5). Section 5 discusses the related literature (Section 5.1), as well as additional features and extensions of the VEU representation (Section 5.2). All proofs, as well as additional technical results, are given in the Appendix. Supplemental material is also available online (Siniscalchi (2009)).

2. NOTATION AND DEFINITIONS

The following notation is standard. Consider a set Ω (the state space) and a sigma-algebra Σ of subsets of Ω (events). It will be useful to assume that the sigma-algebra Σ is *countably generated*: that is, there is a countable collection

[8]A slight modification is required to ensure monotonicity; see Section 4.4 for details.

$\mathcal{S} = (S_i)_{i\geq 0}$ such that Σ is the smallest sigma-algebra containing $\mathcal{S}$. All finite and countably infinite sets, as well as all Borel subsets of Euclidean n-space, and more generally all standard Borel spaces (Kechris (1995)) satisfy this assumption.

Denote by $B_0(\Sigma)$ the set of Σ-measurable real functions with finite range and by $B(\Sigma)$ its sup-norm closure. The set of countably additive probability measures on Σ is denoted by $ca_1(\Sigma)$. For any probability measure $\pi \in ca_1(\Sigma)$ and function $a \in B(\Sigma)$, let $\mathrm{E}_\pi[a] = \int_\Omega a\, d\pi$, the standard Lebesgue integral of a with respect to π. Finally, $a \circ b : \mathcal{X} \to \mathcal{Z}$ denotes the composition of the functions $b : \mathcal{X} \to \mathcal{Y}$ and $a : \mathcal{Y} \to \mathcal{Z}$.

Additional notation is useful to streamline the definition and analysis of the VEU representation. Given $m \in \mathbb{Z}_+ \cup \{\infty\}$ and a finite or countably infinite collection $z = (z_i)_{0\leq i<m}$ of elements of $B(\Sigma)$, let $\mathrm{E}_\pi[z \cdot a] = (\mathrm{E}_\pi[z_i \cdot a])_{0\leq i<m}$ if $m > 0$ and let $\mathrm{E}_\pi[z \cdot a] = 0$ if $m = 0$. For any collection $F \subset B(\Sigma)$, let $\mathcal{E}(F; \pi, z) = \{\mathrm{E}_\pi[z \cdot a] \in \mathbb{R}^m : a \in F\}$. Finally, let 0_m denote the zero vector in $\mathbb{R}^m$.

Turn now to the decision setting. Consider a convex set X of consequences (outcomes, prizes). As in Anscombe and Aumann (1963), X could be the set of finite-support lotteries over some underlying collection of (deterministic) prizes, endowed with the usual mixture operation. Alternatively, the set X might be endowed with a subjective mixture operation, as in Casadesus-Masanell, Klibanoff, and Ozdenoren (2000) or Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003).

An *act* is a Σ-measurable function from Ω to X. Let $\mathcal{F}_0$ be the set of simple acts, that is, acts with finite range. With the usual abuse of notation, denote by x the constant act assigning the consequence $x \in X$ to each $\omega \in \Omega$. The main object of interest is a preference relation $\succcurlyeq$ on $\mathcal{F}_0$; its symmetric and asymmetric parts are denoted $\sim$ and $\succ$, respectively.

As is the case for other decision models, VEU preferences on $\mathcal{F}_0$ have a unique extension to a class of nonsimple, bounded acts. This extension is of particular interest in this paper: Proposition 1 uses it to characterize the minimum number of adjustment factors required to provide a VEU representation of a given preference relation. Thus, following Schmeidler (1989), denote by $\mathcal{F}_b$ the set of acts f for which there exist $x, x' \in X$ such that $x \succcurlyeq f(\omega) \succcurlyeq x'$ for all $\omega \in \Omega$.

Finally, given a function $u : X \to \mathbb{R}$ and a set $\mathcal{F}$ of acts, let $u \circ \mathcal{F} = \{u \circ f \in B(\Sigma) : f \in \mathcal{F}\}$. The formal definition of the VEU representation can now be provided. For the reasons just mentioned, the definition accommodates preferences on either $\mathcal{F}_0$ or $\mathcal{F}_b$.

DEFINITION 1: Let $\mathcal{F}$ denote either $\mathcal{F}_0$ or $\mathcal{F}_b$. A tuple (u, p, n, ζ, A) is a *VEU representation* of a preference relation $\succcurlyeq$ on $\mathcal{F}$ if the following conditions are met:

1. $u : X \to \mathbb{R}$ is nonconstant and affine, $p \in ca_1(\Sigma)$, $n \in \mathbb{Z}_+ \cup \{\infty\}$, and $\zeta = (\zeta_i)_{0\le i<n}$.
2. For every $0 \le i < n$, $\zeta_i \in B(\Sigma)$ and $\mathrm{E}_p[\zeta_i] = 0$.
3. $A : \mathcal{E}(u \circ \mathcal{F}; p, \zeta) \to \mathbb{R}$ satisfies $A(0_n) = 0$ and $A(\varphi) = A(-\varphi)$ for all $\varphi \in \mathcal{E}(u \circ \mathcal{F}; p, \zeta)$.
4. For all $a, b \in u \circ \mathcal{F}$, $a(\omega) \ge b(\omega)$ for all $\omega \in \Omega$ implies $\mathrm{E}_p[a] + A(\mathrm{E}_p[\zeta \cdot a]) \ge \mathrm{E}_p[b] + A(\mathrm{E}_p[\zeta \cdot b])$.
5. For every pair of acts $f, g \in \mathcal{F}$,

$$f \succcurlyeq g \quad\Leftrightarrow\quad \mathrm{E}_p[u\circ f] + A(\mathrm{E}_p[\zeta \cdot u\circ f]) \ge \mathrm{E}_p[u\circ g] + A(\mathrm{E}_p[\zeta \cdot u\circ g]). \tag{3}$$

Conditions 1 and 5 are self-explanatory. Condition 2 ensures that the adjustment factors ζ_i are bounded and reflect the fact that constant acts are not subject to ambiguity. The general representation allows for at most countably infinitely many adjustment factors; moreover, by Theorem 1, if the state space Ω is finite, then a finite n suffices.

In addition to the normalization $A(0_n) = 0$, condition 3 formalizes the central *symmetry* assumption discussed in the Introduction (cf. in particular footnote 7). Condition 4 ensures monotonicity of the VEU representation. Simple examples show that monotonicity necessarily involves a joint restriction on p, ζ, and A.[9] In many cases of interest, easy-to-check necessary and sufficient conditions can be provided: see Appendix A for details.

The functional A can be extended to all of $\mathbb{R}^n$ consistently with the symmetry requirement of condition 3; for instance, let $A(\phi) = 0$ for all $\phi \in \mathbb{R}^n \setminus \mathcal{E}(u \circ \mathcal{F}; p, \zeta)$.[10] The values assumed by A at such points are obviously irrelevant to the representation of preferences. Restricting the domain of A to $\mathcal{E}(u \circ \mathcal{F}; p, \zeta)$ as in Definition 1 simplifies the statement of some results.

It is useful to point out that the functional A, and hence the entire VEU representation, is *not* required to be positively homogeneous. This makes it possible to accommodate, for instance, members of the "variational preferences" family studied by Maccheroni, Marinacci, and Rustichini (2006) that satisfy the key symmetry requirement of this paper; furthermore, it enables differentiable specifications of the adjustment functional A, which would otherwise be precluded.

Observation: *Equivalent Formulations*: One can view the collection $\zeta = (\zeta_i)_{0\le i<n}$ as a *vector-valued function* and view the corresponding n-vector

[9]Refer to the three-color-urn example in the Introduction and let f'_B be a bet that yields 20 dollars if B obtains. Since $A(\varphi) = -|\varphi|$, then $A(\mathrm{E}_p[\zeta_0 \cdot f'_B]) < A(\mathrm{E}_p[\zeta_0 \cdot f_B])$, even though $\mathrm{E}_p[\zeta_0 \cdot f'_B] = \frac{20}{3} > \frac{10}{3} = \mathrm{E}_p[\zeta_0 \cdot f_B]$. Taking $A(\varphi) = |\varphi|$ instead shows that no general assumption may be made regarding the direction of monotonicity for A alone.

[10]Note that $a \in u \circ \mathcal{F}$ implies $[\inf_\Omega a + \sup_\Omega a] - a \in u \circ \mathcal{F}$, so $\phi \in \mathcal{E}(u \circ \mathcal{F}; p, \zeta)$ implies $-\phi \in \mathcal{E}(u \circ \mathcal{F}; p, \zeta)$.

$(\mathrm{E}_p[\zeta_i \cdot u \circ f])_{0\leq i<n}$ as its vector expectation (where integration is in the Bochner sense: cf. Aliprantis and Border (1994, Section 11.8)).[11]

Each adjustment factor ζ_i can also be interpreted as a Radon–Nikodym derivative: that is, one can define a corresponding signed measure $m_i : \Sigma \to \mathbb{R}$ by letting $m_i(E) = \mathrm{E}_p[\zeta_i \cdot 1_E]$ for each $E \in \Sigma$. An earlier version of this paper (Siniscalchi (2007)) employed this formulation.

Finally, it is convenient to define a notion of "parsimonious" VEU representation; the objective is to remove two types of redundancy. First, one or more of the functions $(\zeta_i)_{0\leq i<n}$ in Definition 1 may be linear combinations of other adjustment factors. In the proposed parsimonious VEU representation, the collection $(\zeta_i)_{0\leq i<n}$ is instead required to be *orthonormal* (hence a fortiori linearly independent) relative to the inner product defined by the baseline prior p; that is, for all i, j such that $0 \leq i < n$ and $0 \leq j < n$, $\mathrm{E}_p[\zeta_i\zeta_j] = 1$ if $i = j$ and $\mathrm{E}_p[\zeta_i\zeta_j] = 0$ otherwise. This also suggests that the adjustment factors ζ_i reflect distinct, mutually uncorrelated "sources of ambiguity." The normalization $\mathrm{E}_p[\zeta_i^2] = 0$ is mainly for convenience.

The second type of redundancy is motivated by the decision-theoretic notion of "crisp acts" introduced by Ghirardato, Maccheroni, and Marinacci (2004). Again, let $\mathcal{F}$ denote either $\mathcal{F}_0$ or $\mathcal{F}_b$. Say that an act $f \in \mathcal{F}$ is crisp if, for every $x \in X$ that satisfies $f \sim x$, and for every $g \in \mathcal{F}_0$[12] and $\lambda \in (0, 1]$,

$$\lambda g + (1-\lambda)x \sim \lambda g + (1-\lambda)f. \tag{4}$$

That is, a crisp act behaves like its certainty equivalent: in particular, as discussed in Ghirardato, Maccheroni, and Marinacci (2004), it does not provide a hedge against the ambiguity that influences any other act g.[13] Constant acts are obviously crisp; correspondingly, any VEU representation of the preference $\succcurlyeq$ assigns them the zero adjustment vector. Since crisp acts behave like constant acts, it seems desirable to ensure that their associated adjustment vector also be zero.

DEFINITION 2: Let $\mathcal{F}$ denote either $\mathcal{F}_0$ or $\mathcal{F}_b$. A VEU representation (u, p, n, ζ, A) of a preference relation $\succcurlyeq$ on $\mathcal{F}$ is *sharp* if $(\zeta_i)_{0\leq i<n}$ is orthonormal and, for any crisp act $f \in \mathcal{F}$, $\mathrm{E}_p[\zeta \cdot u \circ f] = 0_n$.

As an immediate implication, note that, for an EU preference, all acts are crisp; thus, the unique sharp VEU representation of an EU preference features $n = 0$, that is, an empty adjustment tuple.

[11]If $n = \infty$, one must normalize the factors ζ_i so that they are uniformly bounded. One then views $\zeta = (\zeta_i)_{0\leq i<\infty}$ as a function with values in the Banach space ℓ_∞.

[12]Under the axioms in the next section, restricting attention to $g \in \mathcal{F}_0$ is without loss even for $f \in \mathcal{F} = \mathcal{F}_b$.

[13]The present definition is weaker than its counterpart in Ghirardato, Maccheroni, and Marinacci (2004): in particular, it allows for preferences that do not have a positively homogeneous representation. The two definitions are equivalent if positive homogeneity holds.

It is sometimes convenient to employ VEU representations that are not sharp: see, for instance, the analysis of updating in Section 4.4. However, adjustment factors in a sharp representation can be interpreted as independent sources of ambiguity; see Section 4 for details.

3. AXIOMATIC CHARACTERIZATION OF VEU PREFERENCES

Mixtures of acts are taken pointwise: for every pair of acts f, g and any $\alpha \in [0,1]$, $\alpha f + (1-\alpha)g$ is the act assigning the consequence $\alpha f(\omega) + (1-\alpha)g(\omega)$ to each state $\omega \in \Omega$.

As in the preceding section, let $\mathcal{F}$ denote either $\mathcal{F}_0$ or $\mathcal{F}_b$. Axioms 1–4 are standard:

AXIOM 1—Weak Order: *$\succcurlyeq$ is transitive and complete on $\mathcal{F}$.*

AXIOM 2—Monotonicity: *For all acts $f, g \in \mathcal{F}$, $f(\omega) \succcurlyeq g(\omega)$ for all $\omega \in \Omega$ implies $f \succcurlyeq g$.*

AXIOM 3—Continuity: *For all acts $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succcurlyeq h\}$ and $\{\alpha \in [0,1] : h \succcurlyeq \alpha f + (1-\alpha)g\}$ are closed.*

AXIOM 4—Nondegeneracy: *Not for all $f, g \in \mathcal{F}$, $f \succcurlyeq g$.*

Next, a weak form of the Anscombe and Aumann (1963) independence axiom, owing to Maccheroni, Marinacci, and Rustichini (2006), is assumed.

AXIOM 5—Weak Certainty Independence: *For all acts $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1)$, $\alpha f + (1-\alpha)x \succcurlyeq \alpha g + (1-\alpha)x$ implies $\alpha f + (1-\alpha)y \succcurlyeq \alpha g + (1-\alpha)y$.*

Loosely speaking, preferences are required to be invariant to translations of utility profiles, but not to rescaling (note that the same weight α is employed when mixing with x and with y). As discussed in Maccheroni, Marinacci, and Rustichini (2006), this axiom weakens Gilboa and Schmeidler's (1989) *certainty independence*, which requires invariance to both translation and rescaling. Since certainty independence will be referenced below, it is reproduced here, even though it is *not* assumed in Theorem 1.

AXIOM 5*—Certainty Independence: *For all acts $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$, $f \succcurlyeq g$ implies $\alpha f + (1-\alpha)x \succcurlyeq \alpha g + (1-\alpha)x$.*

To ensure that the baseline prior is countably additive, adopt the following axiom, which is in the spirit of Arrow (1974).[14] A similar representation could

[14]See also Chateauneuf, Marinacci, Maccheroni, and Tallon (2005) and Ghirardato, Maccheroni, and Marinacci (2004).

be obtained without it, but it would not be possible to restrict attention to finite or countably infinite collections of adjustment factors. To state the axiom, for every pair $x, y \in X$ and $E \in \Sigma$, denote by xEy the act that yields x at every state $\omega \in E$ and y elsewhere.

AXIOM 6—Monotone Continuity: *For all sequences* $(A_k)_{k\geq 1} \subset \Sigma$ *such that* $A_k \supset A_{k+1}$ *and* $\bigcap_k A_k = \emptyset$, *and for all* $x, y, z \in X$ *such that* $x \succ y \succ z$, *there is* $k \geq 1$ *such that* $zA_kx \succ y \succ xA_kz$.

To state the novel axioms in this paper, a preliminary definition is required. Intuitively, it identifies pairs of acts whose utility profiles are "mirror images."

DEFINITION 3: Two acts $f, \bar{f} \in \mathcal{F}$ are *complementary* if and only if, for any two states $\omega, \omega' \in \Omega$,

$$\frac{1}{2}f(\omega) + \frac{1}{2}\bar{f}(\omega) \sim \frac{1}{2}f(\omega') + \frac{1}{2}\bar{f}(\omega').$$

If two acts $f, \bar{f} \in \mathcal{F}$ are complementary, then $(f, \bar{f})$ is referred to as a *complementary pair*.

If preferences over X can be represented by a von Neumann–Morgenstern utility function $u(\cdot)$, which is the case under Axioms 1–5, then the *utility profiles* of the acts f and $\bar{f}$, denoted $u \circ f$ and $u \circ \bar{f}$, respectively, satisfy $u \circ \bar{f} = k - u \circ f$ for some constant $k \in \mathbb{R}$. Thus, complementarity is the preference counterpart of algebraic negation.

Notice that if $(f, \bar{f})$ and $(g, \bar{g})$ are complementary pairs of acts, then, for any weight $\alpha \in [0, 1]$, the mixtures $\alpha f + (1-\alpha)g$ and $\alpha\bar{f} + (1-\alpha)\bar{g}$ are themselves complementary.

The complementary independence axiom may now be formulated.

AXIOM 7—Complementary Independence: *For any two complementary pairs* $(f, \bar{f})$ *and* $(g, \bar{g})$ *in* $\mathcal{F}$, *and all* $\alpha \in [0, 1]$: $f \succcurlyeq \bar{f}$ *and* $g \succcurlyeq \bar{g}$ *imply* $\alpha f + (1-\alpha)g \succcurlyeq \alpha\bar{f} + (1-\alpha)\bar{g}$.

Axiom 7 formalizes the behavioral implications of the key cognitive assumption underlying VEU preferences: the decision-maker's assessment of an act takes into account (i) a baseline evaluation, consistent with EU, as well as (ii) its *utility variability around this baseline*.[15] To elaborate, for EU preferences, the property "$f \succcurlyeq \bar{f}$ and $g \succcurlyeq \bar{g}$ imply that $\alpha f + (1-\alpha)g \succcurlyeq \alpha\bar{f} + (1-\alpha)\bar{g}$" holds regardless of whether or not $f, \bar{f}$ and $g, \bar{g}$ are pairwise complementary;

[15]Equivalently, its *outcome* variability, but taking preferences over prizes into account.

indeed, under Axioms 1–4, this property is equivalent to the standard independence axiom and characterizes EU preferences. Next, recall that complementary acts are mirror images of each other; hence, as noted in the Introduction, virtually all classical measures of dispersion attribute to them the *same utility variability*. Under the cognitive assumptions considered here, this implies that *complementary acts are effectively ranked according to their baseline evaluation*, which is assumed to be consistent with EU. In Axiom 7, this applies to the ranking of f vs. $\bar{f}$, g vs. $\bar{g}$, and, because complementarity is preserved by mixtures, $\alpha f + (1-\alpha)g$ vs. $\alpha\bar{f} + (1-\alpha)\bar{g}$. These rankings must be consistent with EU, which leads to the requirement in Axiom 7.

A final assumption is needed:

AXIOM 8—Complementary Translation Invariance: *For all complementary pairs* $(f, \bar{f})$ *in* $\mathcal{F}$ *and all* $x, \bar{x} \in X$ *with* $f \sim x$ *and* $\bar{f} \sim \bar{x}$, $\frac{1}{2}f + \frac{1}{2}\bar{x} \sim \frac{1}{2}\bar{f} + \frac{1}{2}x$.

Axiom 8 ensures that complementary acts are subject to the same adjustment to their respective baseline evaluations. Observe first that, since f and $\bar{f}$ in Axiom 8 are complementary, so are the mixtures $\frac{1}{2}f + \frac{1}{2}\bar{x}$ and $\frac{1}{2}\bar{f} + \frac{1}{2}x$; hence, these acts are evaluated according to their baseline EU evaluation. Consequently, the indifference between these mixtures has a trade-off interpretation: the difference between the baseline EU evaluation of f and $\bar{f}$ is equal to the utility difference between x and $\bar{x}$. Since $f \sim x$ and $\bar{f} \sim \bar{x}$, it also equals the difference between the overall VEU evaluations of f and $\bar{f}$. Hence, f and $\bar{f}$ are subject to the same adjustment.

Complementary translation invariance is much less central to the characterization of VEU preferences than complementary independence (Axiom 7). Indeed, Axiom 8 is actually redundant in two important cases. First, Axiom 8 is implied by Axioms 1–5 and 7 if the utility function representing preferences over X is unbounded either above or below,[16] as is the case for the majority of monetary utility functions employed in applications. Second, regardless of the utility function, if preferences satisfy Axioms 1–4 and 5* (instead of Axiom 5), then it is trivial to verify that the indifference required by Axiom 8 holds regardless of whether or not f and $\bar{f}$ are complementary; in other words, Axiom 8 is automatically satisfied by all "invariant-biseparable" preferences Ghirardato, Maccheroni, and Marinacci (2004).[17] Thus, Axiom 8 is *only* required to allow for preferences that simultaneously violate Axiom 5* and are represented by a bounded utility function on X.

The main result of this paper can now be stated.

[16]A proof is available upon request. Well known axioms ensure that utility is unbounded; see, for example, Maccheroni, Marinacci, and Rustichini (2006).

[17]This class includes, for instance, all multiple-priors, α-maxmin, and Choquet expected utility preferences.

THEOREM 1: *Consider a preference relation $\succcurlyeq$ on $\mathcal{F}_0$. The following statements are equivalent*:

1. *The preference relation $\succcurlyeq$ satisfies Axioms 1–8 on $L = \mathcal{F}_0$.*
2. *$\succcurlyeq$ admits a sharp VEU representation (u, p, n, ζ, A).*
3. *$\succcurlyeq$ admits a VEU representation (u, p, n, ζ, A).*

In statement 2, if (u', p', n', ζ', A') is another VEU representation of $\succcurlyeq$, then $p' = p$, $u' = \alpha u + \beta$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$, and there is a linear surjection $T : \mathcal{E}(u' \circ \mathcal{F}_0; p, \zeta') \to \mathcal{E}(u \circ \mathcal{F}_0; p, \zeta)$ such that

$$\forall a' \in u' \circ \mathcal{F}_0, \quad T(\mathrm{E}_p[\zeta' \cdot a']) = \frac{1}{\alpha}\mathrm{E}_p[\zeta \cdot a'], \tag{5}$$
$$A'(\mathrm{E}_p[\zeta' \cdot a']) = \alpha A\big(T(\mathrm{E}_p[\zeta' \cdot a'])\big).$$

If (p, u', n', ζ', A') is also sharp, then T is a bijection. Finally, if Ω is finite, then $n \le |\Omega| - 1$.

COROLLARY 1: *If a preference relation on $\mathcal{F}_0$ satisfies satisfies Axioms 1–8, then it has a unique extension to $\mathcal{F}_b$ that satisfies the same axioms and admits a sharp VEU representation on $\mathcal{F}_b$.*

The primary message of Theorem 1 is the equivalence of statements 1 and 2: Axioms 1–8 are equivalent to the existence of a *sharp* VEU representation. However, as noted in Section 2, it is sometimes convenient to employ VEU representations that are not sharp. Theorem 1 ensures that the resulting preferences will still satisfy Axioms 1–8. To put it differently, if a preference admits a VEU representation, then it also admits a sharp VEU representation.

The second part of Theorem 1 indicates the uniqueness properties of the VEU representation. The baseline probability measure p is unique, and the adjustment factors ζ and function A are unique up to transformations that preserve both the affine structure of the set $\mathcal{E}(u \circ \mathcal{F}_0; p, \zeta)$ of adjustment vectors and the actual adjustment associated with each element in that set.

To elaborate, recall that the role of the adjustment factors ζ is to capture the patterns of "complementarity" among different events; for instance, if ambiguity about two events E and F cancels out, then $\mathrm{E}_p[\zeta \cdot 1_{E\cup F}] = 0$. For another tuple of random variables ζ' to capture the same complementarities as ζ, it must be the case that also $\mathrm{E}_p[\zeta' \cdot 1_{E\cup F}] = 0$. Similarly, complementarities among adjustment vectors associated with different acts must be preserved. The existence of a functional T with the properties listed in Theorem 1 ensures this. As Example 1 illustrates, this imposes considerable restrictions on transformations of a given adjustment that can be deemed inessential.

EXAMPLE 1: Refer to the ambiguity-averse VEU preferences described in the Introduction in the context of the Ellsberg paradox. Note that $\mathcal{E}(u \circ \mathcal{F}_0; p, \zeta)$ is the entire real line.

Now consider a two-element tuple $\zeta' = (\zeta'_0, \zeta'_1)$ and let $A'(\varphi) = -\sqrt{\varphi_1^2 + \varphi_2^2}$ for all $\varphi \in \mathcal{E}(u \circ \mathcal{F}_0; p, \zeta')$. Suppose T is as in Theorem 1. Then $A' = A \circ T$ implies that, in particular, $A'(\frac{1}{3}\zeta'(R)) = A(T(\frac{1}{3}\zeta'(R))) = A(\frac{1}{3}\zeta(R)) = 0$, so $\zeta'(R) = 0 \in \mathbb{R}^2$. Similarly, $T(\frac{1}{3}\zeta'(B) + \frac{1}{3}\zeta'(G)) = \frac{1}{3}\zeta(B) + \frac{1}{3}\zeta(G) = 0$, so $A' = A \circ T$ implies $A'(\frac{1}{3}\zeta'(B) + \frac{1}{3}\zeta'(G)) = 0$, and so $\zeta'(B) = -\zeta'(G)$. Finally, $A'(\frac{1}{3}\zeta'(B)) = \frac{1}{3} = A'(\frac{1}{3}\zeta'(G))$. In other words, ζ' encodes exactly the same information about B and G as ζ: the two events are equally ambiguous, but their ambiguities cancel out. Of course, ζ does so in a more parsimonious way. Thus, intuitively, ambiguity in the Ellsberg paradox is really "one dimensional," regardless of the particular vector representation one chooses. The analysis in Section 4.1 expands upon this observation.

4. ANALYSIS OF THE REPRESENTATION AND ADDITIONAL RESULTS

4.1. *Heuristic Construction of the Representation*

The VEU representation is constructed in three key steps. First, a preliminary numerical representation is obtained invoking results from Maccheroni, Marinacci, and Rustichini (2006); see item 6 in Proposition 6. Second, the baseline prior p is identified: Lemma 1 (cf. also Observation 1) implies that if Axioms 7 and 8 hold,[18] there exists a unique probability p such that, for every complementary pair $(f, \bar{f})$, $f \succcurlyeq \bar{f}$ iff $\mathrm{E}_p[u \circ f] \geq \mathrm{E}_p[u \circ \bar{f}]$, as was claimed in the Introduction. By Axiom 6, p is countably additive (Lemma 5). The third key step is the construction of the adjustment factors ζ_i and the function A. To provide some intuition, it is useful to focus once again on the three-color-urn problem of the Introduction and Example 1.

Recall that the prior p on the state space $\Omega = \{R, G, B\}$ is assumed to be uniform. Figure 1 depicts the set $\mathcal{F}_0$ of acts in the problem under consideration; assuming linear utility for simplicity, this is identified with Euclidean space $\mathbb{R}^3$. The upward-sloping plane in the picture corresponds to the set of crisp acts; in this example, ambiguity concerns the relative likelihood of G vs. B, so intuitively an act h is crisp if and only if $h(G) = h(B)$. Denote this set by C and denote by NC the *orthogonal complement* of C relative to the inner product defined by the baseline prior p: that is, $g \in NC$ if and only if $\mathrm{E}_p[g \cdot h] = 0$ for all $h \in C$. In Figure 1, this set corresponds to the line perpendicular to C and going through the origin.[19] By definition, elements of NC are *uncorrelated with any crisp act*, and thus may be thought of as "purely ambiguous"; the acronym NC stands for the more neutral term *noncrisp*. In this example, both C and NC

[18]As noted above, Axiom 8 need not be imposed explicitly in most cases of interest for applications.

[19]Since p is uniform, in this example the elements of NC are also orthogonal to C in the usual Euclidean sense.

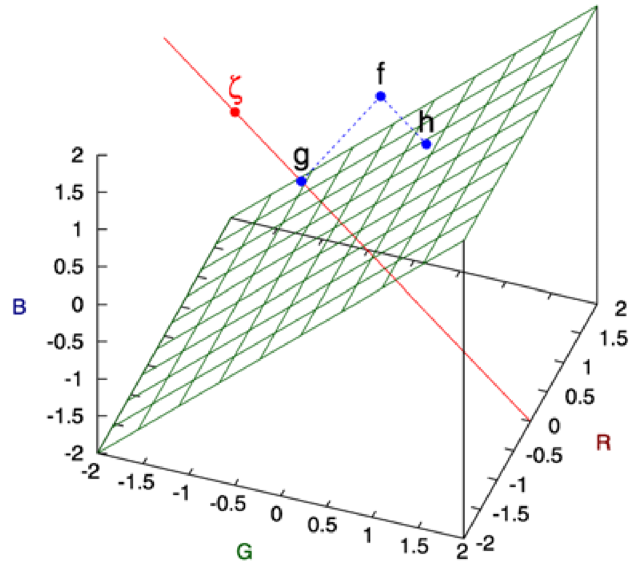

FIGURE 1.—Crisp and noncrisp acts in the Ellsberg paradox.

are easily seen to be closed subsets of $\mathbb{R}^3$. For the general case, see Lemma 6 in the Appendix.

It is now possible to define the collection $\zeta = (\zeta_i)_{0 \leq i < n}$ as an *orthonormal basis for the set NC*. In this example, NC is one dimensional (recall Example 1), so ζ consists of a single vector; Figure 1 depicts one of only two possible choices for ζ (the other is the negative of the vector indicated in the picture). Also observe that, because it must lie on NC, $\zeta = (\zeta_0)$ must necessarily satisfy $\zeta_0(G) = -\zeta_0(B)$. Thus, the key feature of the adjustment factor used to rationalize the modal preferences in the Ellsberg paradox actually arises *endogenously* from this construction, once the set of crisp acts has been specified. The existence of an orthonormal basis in the general case is a standard property of Hilbert spaces; furthermore, under the assumption that the sigma-algebra Σ is countably generated, such a basis is countable.

Finally, consider an act f: its projections g and h onto NC and C, respectively, are uniquely defined; this is immediate in the example, and follows from the orthogonal decomposition theorem (cf., e.g., Dudley (1989, p. 125)) in the general case. One can thus think of g and h as the purely ambiguous and crisp parts of the act f. This decomposition has two useful consequences.

First, it can be shown that the difference between the individual's evaluation (equivalently, due to the assumption of linear utility, the certainty equivalent) of the act f and its baseline expectation $\mathrm{E}_p[f]$ depends *solely* upon the projection g of f on NC—that is, solely on its ambiguous part. Second, the projection of f on NC has a representation in terms of the adjustment factor ζ_0: in particular, $g = \mathrm{E}_p[\zeta_0 \cdot f] \cdot \zeta_0$. In the general case, the expectations $\mathrm{E}_p[\zeta_i \cdot u \circ f]$, viewed as inner products, are the *Fourier coefficients* of f relative to the orthonormal basis $\zeta = (\zeta_i)_{0 \le i < n}$ of the Hilbert space NC. Taken together, these facts lead to the VEU representation in Eq. (3).

4.2. *Characterization of the Number n of Adjustment Factors*

The cardinality n of the orthonormal basis ζ has a direct behavioral characterization. A notion of "linear combination" of acts, that is, a "mixture" that allows for negative weights, is required. Complementarity (Definition 3) enables a straightforward formulation of this notion: a *combination* of a collection of acts $f_1, \dots, f_m \in \mathcal{F}_b$ is a mixture act $\alpha_1 g_1 + \cdots + \alpha_m g_m \in \mathcal{F}_b$, where $\sum_i \alpha_i = 1$ and, for every $i = 1, \dots, m$, $\alpha_i \in [0, 1]$ and either $g_i = f_i$ or g_i is complementary to f_i.

PROPOSITION 1: *Consider a preference relation $\succcurlyeq$ on $\mathcal{F}_0$ that satisfies Axioms 1–4 and let (u, p, n, ζ, A) be a VEU representation of its unique extension to $\mathcal{F}_b$.*

1. *For every finite $m > n$, every tuple $f_1, \dots, f_m \in \mathcal{F}_b$ admits a crisp combination.*

If, additionally, (u, p, n, ζ, A) is sharp, then

2. *for every finite $m \le n$, there is a tuple $f_1, \dots, f_m \in \mathcal{F}_b$ that admits no crisp combination;*
3. *for every other VEU representation (u', p', n', ζ', A') of the extension of $\succcurlyeq$ to $\mathcal{F}_b$, $n' \ge n$.*
4. *$n = 1$ if and only if $\succcurlyeq$ is not consistent with EU and, for all $f, g, \bar{g} \in \mathcal{F}_b$ such that $g, \bar{g}$ are complementary and not constant, and all $\alpha \in [0, 1]$, either $\alpha f + (1 - \alpha) g$ or $\alpha f + (1 - \alpha) \bar{g}$ is crisp.*

This result complements the analysis in the preceding subsection, and reinforces the interpretation of the number n as reflecting the multiplicity and complexity of the "sources of ambiguity" in a given decision situation. Part 1 of Proposition 1 states that, given any collection of more than n acts, it is possible to construct a crisp combination, that is, a perfect hedge against ambiguity. Intuitively, this means that there cannot be more than n distinct sources or forms of ambiguity; for instance, in the three-color-urn example, given any two noncrisp acts, it is always possible to construct a combination act that delivers the same outcome in states G and B, and is therefore not subject to ambiguity. Conversely, part 2 of the Proposition 1 asserts the existence of a tuple of up to n acts that cannot be combined in any way to construct a perfect hedge. Intuitively, this suggests that each act in such a tuple is subject to a different source or form of ambiguity. It is also instructive to note that the tuple $f_1, \dots, f_m$ in the statement is constructed by rescaling the adjustment factors $(\zeta_i)_{0\le i<n}$.[20]

Part 3 of Proposition 1 complements the uniqueness statement of Theorem 1. Consider a sharp VEU representation (u, p, n, ζ, A) of the extension of $\succcurlyeq$ to $\mathcal{F}_b$. A fortiori, this[21] is a sharp VEU representation of $\succcurlyeq$ on $\mathcal{F}_0$, and Theorem 1 states that (u, p, n, ζ, A) employs the "smallest" set of adjustment vectors $\mathcal{E}(u \circ \mathcal{F}_0; p, \zeta)$, up to embedding. Proposition 1 additionally ensures that the sharp representation (u, p, n, ζ, A) employs the minimal number of adjustment factors.

4.3. *The Adjustment Function A and Ambiguity Attitudes*

This section analyzes ambiguity aversion for VEU preferences. Two established definitions of this concept are considered: Schmeidler (1989) and Ghirardato and Marinacci (2002). Both have natural characterizations in terms of properties of the adjustment function A. Ghirardato and Marinacci's notion also allows for comparisons of ambiguity attitudes across individuals: again, a characterization in terms of the adjustment function A is provided.

Begin with Schmeidler's classical axiom. Intuitively, an individual who is ambiguity-averse according to the proposed definition values mixtures because they "*smooth*" *utility profiles* (cf. Schmeidler (1989, p. 582), Klibanoff (2001, p. 290)). This has a straightforward characterization for VEU preferences, stated below as a corollary to Theorem 1.

AXIOM 9—Ambiguity Aversion: *For all $f, g \in \mathcal{F}_0$ and $\alpha \in (0,1)$, $f \sim g$ implies $\alpha f + (1-\alpha)g \succcurlyeq g$.*

COROLLARY 2: *Consider a preference relation $\succcurlyeq$ on $\mathcal{F}_0$ for which Axioms 1–8 hold and let A be as in Theorem 1 statement 2. Then $\succcurlyeq$ satisfies Axiom 9 if and only if A is nonpositive and concave.*

A VEU preference that satisfies Axiom 9 is variational (Maccheroni, Marinacci, and Rustichini (2006)); if it additionally satisfies certainty independence (Axiom 5*) rather than the weaker Axiom 5, then it is a maxmin EU preference (Gilboa and Schmeidler (1989)). For completeness, a VEU preference is *ambiguity-loving* in the sense of Schmeidler (i.e., $f \sim g$ implies $\alpha f + (1-\alpha)g \preccurlyeq g$ for all $f, g \in \mathcal{F}_0$ and $\alpha \in (0,1)$) if and only if A is nonnegative and convex; it is *ambiguity-neutral* (i.e., both ambiguity-averse and ambiguity-loving) if and only if $A = 0$.

[20]This is the reason why the extension of $\succcurlyeq$ to $\mathcal{F}_b$ is required. To the best of my knowledge, one cannot guarantee that the adjustment factors are simple functions, although they can be shown to be bounded.

[21]Strictly speaking, consider (u, p, n, ζ, A_0), where A_0 is the restriction of A to $\mathcal{E}(u \circ \mathcal{F}_0; p, \zeta)$.

In the VEU representation, it also seems plausible to associate nonpositive, but not necessarily concave, adjustment functions with a (different) form of ambiguity aversion. This property turns out to be characterized by weaker forms of Axiom 9 for VEU preferences.

AXIOM 10—Complementary Ambiguity Aversion: *For all complementary pairs* $(f, \bar{f})$ *and prizes* $x, \bar{x} \in X$ *such that* $f \sim x$ *and* $\bar{f} \sim \bar{x}$, $\frac{1}{2}f + \frac{1}{2}\bar{f} \succcurlyeq \frac{1}{2}x + \frac{1}{2}\bar{x}$.

AXIOM 11—Simple Diversification: *For all complementary pairs* $(f, \bar{f})$ *with* $f \sim \bar{f}$, $\frac{1}{2}f + \frac{1}{2}\bar{f} \succcurlyeq f$.

Both axioms have the standard hedging interpretation, but are restricted to complementary acts. Axiom 11 is related to the "diversification" property of Chateauneuf and Tallon (2002).

Finally, Ghirardato and Marinacci (2002) proposed a way to compare ambiguity attitudes across decision-makers, mirroring analogous definitions for risk attitudes. This leads to a "comparative" notion of ambiguity aversion. For VEU preferences, this notion, too, characterizes a negative adjustment function. The details are as follows.

DEFINITION 4: Given two preference relations $\succcurlyeq_1$ and $\succcurlyeq_2$ on $\mathcal{F}_0$, $\succcurlyeq_1$ is *more ambiguity-averse than* $\succcurlyeq_2$ iff, for all $f \in \mathcal{F}_0$ and $x \in X$, $f \succcurlyeq_1 x \Rightarrow f \succcurlyeq_2 x$. Also, $\succcurlyeq_1$ is *comparatively ambiguity-averse* if it is more ambiguity-averse than a preference relation $\succcurlyeq_2$ that is consistent with EU.

PROPOSITION 2: *Let* $\succcurlyeq$ *be a preference relation with VEU representation* (u, p, n, ζ, A). *Then the following statements are equivalent*:

1. $\succcurlyeq$ *is comparatively ambiguity-averse*.
2. $\succcurlyeq$ *satisfies Axiom* 10.
3. *For all* $\varphi \in \mathcal{E}(u \circ \mathcal{F}_0; p, \zeta)$, $A(\varphi) \leq 0$.

If $u(X)$ *is unbounded above or below*, *or if* $\succcurlyeq$ *satisfies Axiom* 5*, *then statements* 1–3 *are equivalent to the following statement*:

4. $\succcurlyeq$ *satisfies Axiom* 11.

A VEU preference that satisfies the equivalent conditions 1–4 is *not* necessarily variational or, a fortiori, consistent with maxmin EU. (For completeness, such a VEU preference is also *not* ambiguity-loving in the sense of Schmeidler, except in the trivial case, i.e., if it is ambiguity-neutral.) The following example shows that this additional flexibility can be advantageous.

EXAMPLE 2: Machina (2009) considered the following situation. Let $\Omega = \{\omega_1, \dots, \omega_4\}$ and assume that $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$ are known to be equally likely (and not ambiguous); the relative likelihood of ω_1 vs. ω_2 and of ω_3 vs.

TABLE I

MACHINA'S REFLECTION EXAMPLE: REASONABLE PREFERENCES $f_1 \prec f_2$ AND $f_3 \succ f_4$

	ω_1	ω_2	ω_3	ω_4
f_1	\$4000	\$8000	\$4000	\$0
f_2	\$4000	\$4000	\$8000	\$0
f_3	\$0	\$8000	\$4000	\$4000
f_4	\$0	\$4000	\$8000	\$4000

ω_4, is not known. Assume further that $X = \mathbb{R}$ and u is linear (this is inconsequential for the example). Consider the monetary bets (acts) in Table I.

Notice that f_1 and f_4 only differ by a "reflection," that is, by exchanging prizes on states that are *informationally symmetric*. The same is true of f_2 and f_3. Hence, it is plausible to expect that $f_1 \sim f_4$ and $f_2 \sim f_3$. In particular, Machina (2009) conjectured, and L'Haridon and Placido (2009) verified experimentally, that a plausible pattern of "ambiguity-averse" preferences is $f_1 \prec f_2$ and $f_3 \succ f_4$. Machina showed that this pattern is inconsistent with Choquet EU if informational symmetries are respected. Baillon, L'Haridon, and Placido (2008) showed that the same is true for maxmin EU and variational preferences. Recall that the latter two preference models satisfy Schmeidler's notion of ambiguity aversion.[22]

However, it is possible to rationalize this pattern with VEU preferences that satisfy comparative ambiguity aversion and respect informational symmetries. Let p be uniform and define two adjustment factors by $\zeta_0(\omega_1) = 1 = -\zeta_0(\omega_2)$, $\zeta_1(\omega_3) = 1 = -\zeta_1(\omega_4)$, and $\zeta_0(\omega_3) = \zeta_0(\omega_4) = \zeta_1(\omega_1) = \zeta_1(\omega_2) = 0$. Finally, consider the adjustment function $A : \mathbb{R}^2 \to \mathbb{R}$ given by $A(\phi_0, \phi_1) = -\frac{1}{2}\sqrt{1 + |\phi_0|} - \frac{1}{2}\sqrt{1 + |\phi_1|} + 1$. Monotonicity may be verified by applying Remark 2 (Appendix); straightforward calculations show that the pattern $f_1 \prec f_2$ and $f_3 \succ f_4$ is obtained. Finally, $A(\phi_0, \phi_1) \leq 0$ for all (ϕ_0, ϕ_1), and so these VEU preferences are comparatively ambiguity-averse by Proposition 2. Since the adjustment function A is not concave on $\mathbb{R}^2$, these VEU preferences do not satisfy Axiom 9 and hence are not variational, and since $A(\phi) < 0$ unless $\phi = 0$ and A is not convex, these VEU preferences are also not ambiguity-loving.

For additional discussion of Machina's reflection example, see Siniscalchi (2008).

Turn now to the comparison of ambiguity attitudes across individuals. The Ghirardato and Marinacci "more ambiguity averse than" ordering also has a

[22]Smooth-ambiguity preferences Klibanoff, Marinacci, and Mukerji (2005) also rule out this pattern under the appropriate ambiguity-aversion assumption (concavity of the second-order utility).

simple characterization for VEU preferences. To obtain a meaningful comparison of ambiguity attitudes, it is necessary to ensure that the preferences being compared are represented by the same utility function and baseline prior.[23] Furthermore, a comparison solely in terms of the adjustment functions can be obtained if the preferences under consideration also share the same adjustment factors. Proposition 3 provides behavioral characterizations of these conditions, and Proposition 4 characterizes the "more ambiguity averse than" relation for the VEU representation.

PROPOSITION 3: *Consider two VEU preferences* $\succcurlyeq_1$ *and* $\succcurlyeq_2$ *with representations* $(u^1, p^1, n^1, \zeta^1, A^1)$ *and* $(u^2, p^2, n^2, \zeta^2, A^2)$. *Then the following statements are equivalent*:

1. *For all complementary pairs* $(f, \bar{f})$ *in* $\mathcal{F}_0$, $f \succcurlyeq_1 \bar{f}$ *if and only if* $f \succcurlyeq_2 \bar{f}$.
2. $p^1 = p^2$ *and* u^1, u^2 *differ by a positive linear transformation*.

Furthermore, if statement 1 *holds, then* $\succcurlyeq_1$ *and* $\succcurlyeq_2$ *admit a sharp VEU representation with the same vector of adjustment factors if and only if they admit the same set of crisp acts*.[24]

PROPOSITION 4: *Consider two VEU preferences* $\succcurlyeq_1$ *and* $\succcurlyeq_2$ *on* $\mathcal{F}_0$ *with representations* $(u, p, n^1, \zeta^1, A^1)$ *and* $(u, p, n^2, \zeta^2, A^2)$. *Then* $\succcurlyeq_1$ *is more ambiguity-averse than* $\succcurlyeq_2$ *if and only if, for all* $f \in \mathcal{F}_0$, $A^1(\mathrm{E}_p[\zeta^1 \cdot u \circ f]) \leq A^2(\mathrm{E}_p[\zeta^2 \cdot u \circ f])$. *In particular, if* $n^1 = n^2$ *and* $\zeta^1 = \zeta^2 = \zeta$, *then* $\succcurlyeq_1$ *is more ambiguity-averse than* $\succcurlyeq_2$ *if and only if* $A^1(\varphi) \leq A^2(\varphi)$ *for all* $\varphi \in \mathcal{E}(u \circ \mathcal{F}_0; p, \zeta)$.

To conclude, Epstein (1999) proposed an alternative definition of ambiguity aversion in which the benchmark is probabilistic sophistication (Machina and Schmeidler (1992)) rather than EU. The implications of this definition for VEU preferences are left to future work.

4.4. *Updating*

This section proposes an updating rule for VEU preferences. Throughout this subsection, two binary relations on $\mathcal{F}_0$ will be considered: $\succcurlyeq$ denotes the individual's *ex ante* preferences, whereas $\succcurlyeq_E$ denotes her *preferences conditional upon the event* $E \in \Sigma$. To keep notation to a minimum, the event E will be fixed throughout.

To provide some heuristics for the proposed updating rule, recall that the VEU preference functional $V : \mathcal{F}_0 \to \mathbb{R}$ can be rewritten in "covariance" form: compare Eq. (2) in the Introduction. One possible way the individual might

[23]Note that the ranking in Definition 4 already implies that the utility functions coincide: see the proof of Proposition 2.

[24]The final statement is *not* true for VEU representations that are not sharp: examples are readily obtained.

update her preferences upon learning that the event E has occurred is to *update her baseline prior p and use the same functional representation*: that is, replace $\mathrm{E}_p[\cdot]$ and $\mathrm{Cov}_p(\cdot,\cdot)$ in Eq. (2) with $\mathrm{E}_p[\cdot|E]$ and $\mathrm{Cov}_p(\cdot,\cdot|E)$, where $\mathrm{Cov}_p(a,b|E) = \mathrm{E}_p[(a-\mathrm{E}_p[a|E])(b-\mathrm{E}_p[b|E])|E]$.[25] However, the resulting preferences may violate monotonicity, and in fact the functional A may not even be defined for all vectors $(\mathrm{Cov}_p(\zeta_i, u\circ f|E))_{0\le i<n}$. Now consider rescaling conditional covariances by the factor $p(E)$, which leads to

$$V_E(f) = \mathrm{E}_p[u\circ f|E] + A\big((p(E)\cdot \mathrm{Cov}_p(\zeta_i, u\circ f|E))_{0\le i<n}\big). \tag{6}$$

Note that, for $E=\Omega$, the above equation reduces to Eq. (2) in the Introduction. Proposition 5 below shows that Eq. (6) *does* define a well posed VEU, hence monotonic, representation and admits a straightforward behavioral characterization. Observe that Eq. (6) may be equivalently rewritten similarly to Eq. (3) by defining suitable *conditional adjustment factors*:

$$V_E(f) = \mathrm{E}_p[u\circ f|E] + A\big((\mathrm{E}_p(\zeta_{i,E}\cdot u\circ f|E))_{0\le i<n}\big), \tag{7}$$
$$\text{where} \quad \zeta_{i,E} = p(E)\cdot[\zeta_i - \mathrm{E}_p[\zeta_i|E]].$$

Turn now to the axiomatic analysis. The following standard requirement ensures that the conditioning event E "matters" for the individual, so that updating is well defined:

AXIOM 12—E Is Not Null: *There exist $f, g \in \mathcal{F}_0$ such that $f(\omega) = g(\omega)$ for all $\omega \notin E$ and $f \succ g$.*

REMARK 1: *Let $\succcurlyeq$ be a VEU preference, with baseline prior p. Then Axiom 12 holds iff $p(E) > 0$.*

As is the case for conditional EU preferences, it will be assumed throughout that the evaluation of acts upon learning that the event E has occurred does *not* depend upon the consequences that might have been obtained if, counterfactually, E had not obtained:

AXIOM 13—Null Complement: *For all $f, g \in \mathcal{F}_0$, if $f(\omega) = g(\omega)$ for all $\omega \in E$, then $f \sim_E g$.*

The main axiom of this section can be informally stated as follows: *if two acts have the same baseline EU evaluation both ex ante and conditional upon E, and the utility of the outcomes they deliver differs from this baseline evaluation only on the event E, then their ex ante and conditional ranking should be the same*. This is consistent with the proposed interpretation of VEU preferences. Consider

[25]In the covariance formulation, the fact that, in general, $\mathrm{E}_p[\zeta_i|E] \neq 0$ is inconsequential.

an individual whose preferences are VEU both ex ante and conditional on E. Upon learning that E has occurred, her evaluation of an act f may change for two reasons: the baseline EU evaluation of f may change, and utility variability in states outside E no longer matters. However, for acts such that the baseline evaluation does *not* change upon conditioning on E, and which exhibit *no* variation away from the baseline evalution at states outside E to begin with, it seems plausible to assume that the individual's evaluation of such acts will not change.

These special acts can be characterized by a behavioral condition that, once again, involves complementarity. Consider two complementary acts $h, \bar{h} \in \mathcal{F}_0$ that are *constant on* $\Omega \setminus E$; that is, $h(\omega) = h(\omega')$ and $\bar{h}(\omega) = \bar{h}(\omega')$ for all $\omega, \omega' \in \Omega \setminus E$. Suppose that, for any (hence all) $\omega \in \Omega \setminus E$,

$$\frac{1}{2}h + \frac{1}{2}\bar{h}(\omega) \sim \frac{1}{2}\bar{h} + \frac{1}{2}h(\omega). \tag{8}$$

If the preference relation $\succcurlyeq$ happens to be consistent with EU, then Eq. (8), together with complementarity, readily imply that $h \sim h(\omega)$ for any (hence all) $\omega \in \Omega \setminus E$.[26] This indicates that $h(\omega)$ is a certainty equivalent of h ex ante. However, intuitively, $h(\omega)$ can also be viewed as a "conditional certainty equivalent" of h given E: since $h(\omega') = h(\omega)$ for all $\omega' \in \Omega \setminus E$, the ranking $h \sim h(\omega)$ suggests that receiving $h(\omega)$ for sure at states in E is just as good for the individual as allowing the act h to determine the ultimate prize she will receive conditional upon E.[27] Thus, for an EU preference, Eq. (8) implies that the act h has the same certainty equivalent both ex ante and conditional upon E.

For general VEU preferences, the above intuition obviously does not apply: it may well be the case that $h \nsim h(\omega)$ for $\omega \in \Omega \setminus E$. However, recall that complementary independence (Axiom 7) implies that *VEU preferences always rank complementary acts in accordance with their baseline EU evaluation*. Since the mixture acts in Eq. (8) are complementary, the above intuition *does apply* to the EU preference determined by the individual's baseline prior. One then concludes that if Eq. (8) holds, then $h(\omega)$ is a *baseline certainty equivalent* of h, both ex ante and conditional upon E; this is formally verified in the proof of Proposition 5. Furthermore, it is clear that h deviates from this baseline only at states in E. Thus, Eq. (8) identifies the class of acts that should be ranked consistently by prior and conditional VEU preferences.

[26] By complementarity, $\frac{1}{2}h + \frac{1}{2}\bar{h} \sim \frac{1}{2}h(\omega) + \frac{1}{2}\bar{h}(\omega)$; by independence, combining this relation with Eq. (8) yields $\frac{1}{2}h + \frac{1}{2}k \sim \frac{1}{2}h(\omega) + \frac{1}{2}k$, with $k = \frac{1}{2}\bar{h} + \frac{1}{2}\bar{h}(\omega)$. Invoking independence once more yields $h(\omega) \sim h$.

[27] Indeed, this condition may be used to characterize Bayesian updating for EU preferences, as well as prior-by-prior Bayesian updating for maxmin expected utility (MEU) preferences; see Pires (2002).

AXIOM 14—Baseline-Variation Consistency: *For all complementary pairs* $(f, \bar{f})$ *and* $(g, \bar{g})$ *such that* $f, \bar{f}, g, \bar{g}$ *are constant on* $\Omega \setminus E$, *and for every* $\omega \in \Omega \setminus E$, $\frac{1}{2}f + \frac{1}{2}\bar{f}(\omega) \sim \frac{1}{2}\bar{f} + \frac{1}{2}f(\omega)$ *and* $\frac{1}{2}g + \frac{1}{2}\bar{g}(\omega) \sim \frac{1}{2}\bar{g} + \frac{1}{2}g(\omega)$, $f \succcurlyeq_E g$ *if and only if* $f \succcurlyeq g$.

PROPOSITION 5: *Consider a preference relation* $\succcurlyeq$ *on* $\mathcal{F}_0$ *having a VEU representation* (u, p, n, ζ, A), *an event* $E \in \Sigma$, *and another binary relation* $\succcurlyeq_E$ *on* $\mathcal{F}_0$. *Assume that* $\succcurlyeq_E$ *is complete and transitive, and that Axiom* 12 *holds. Then the following statements are equivalent.*

1. *Axioms* 13 *and* 14 *hold.*
2. $\succcurlyeq_E$ *has a VEU representation* $(u, p(\cdot|E), n, \zeta_E, A)$, *where* $\zeta_E = (\zeta_{i,E})_{0\le i<n}$ *is as in Eq.* (7).

It should be noted that the resulting VEU representation is not necessarily sharp, even if the ex ante representation is. Also observe that the updating rule for adjustment factors in Eq. (7) satisfies a version of the "law of iterated conditioning." Fix two events $E, F \in \Sigma$ with $E \subset F$ and, for all $0 \le i < n$, let $\zeta_{i,E,F}$ be the adjustment factor obtained from $\zeta_{i,E}$ by applying Eq. (7), with $p(\cdot|E)$ and $\zeta_{i,E}$ in lieu of p and ζ_i. Then $\zeta_{i,E,F} = \zeta_{i,F}$ for all indices i. Therefore, conditioning on E first, then conditioning the resulting adjustment factors on F yields the same tuple of adjustment factors as conditioning on F directly. This property is shared by some, but not all updating rules for known decision models under ambiguity: for instance, the "maximum-likelihood" rule for maxmin EU preferences (Gilboa and Schmeidler (1993)) violates it.

4.5. *Recursion: A Consumption–Savings Example*

The conditional preferences derived in Proposition 5 only satisfy a weak form of dynamic consistency. Thus, a criterion such as *consistent planning* (Strotz (1955–1956)) is required to resolve possible conflicts between the ex ante and ex post evaluation of future choices. However, the updating rule axiomatized in Section 4.4 allows for a *recursive* formulation of the consistent-planning problem. This section illustrates the basic idea by means of a simple example.

As a preliminary step, it is immediate to verify that if $\Pi \subset \Sigma$ is a finite partition of Ω and if, for every $F \in \Pi$, the tuple $(\zeta_{i,F})_{0\le i<n}$ is defined as in Eq. (7), then

$$\mathrm{E}_p[\zeta_i a] = \sum_{F\in\Pi} \mathrm{E}_p[\zeta_{i,F} a|F] + \sum_{F\in\Pi} p(F)\mathrm{E}_p[\zeta_i|F]\mathrm{E}_p[a|F]. \tag{9}$$

In other words, for every i, the coefficient $\mathrm{E}_p[\zeta_i a]$ can be obtained from the conditional baseline expectations $\mathrm{E}_p[a|F]$ and conditional coefficients $\mathrm{E}_p[\zeta_{i,F} a|F]$ for all $F \in \Pi$, just like the baseline expectation $\mathrm{E}_p[a]$ can be obtained from the conditional baseline expectations $\mathrm{E}_p[a|F]$.

Turn now to the consumption–savings example.

Setup and Notation

Consider an agent who has an initial endowment, or wealth, of w_0 units of a single good and wishes to consume in periods $t = 0, \dots, T$. At each time $t = 0, \dots, T-1$, she can choose how much of her current wealth w_t to save (s_t) and to consume ($c_t = w_t - s_t$). A unit saved at time t yields r_t units of the good at time $t+1$, where $(r_t)_{0\le i<T}$ is an independent and identically distributed (i.i.d.) collection of random variables and each r_t equals either $H > 1$ or $L < H$ with equal probability. This is the only technology that allows the agent to transfer the good across periods. Informally, I shall assume that the agent perceives ambiguity about the *correlation* between r_t and r_{t+1}; this is inspired by Seidenfeld and Wasserman (1993).

Formally, let the state space Ω be the collection of all realizations of the process $(r_t)_{0\le t<T}$, and represent information by a filtration $(\Pi_t)_{0\le t\le T}$, where Π_t is the partition of Ω generated by $r_0, \dots, r_{t-1}$ (so $\Pi_0 = \{\Omega\}$). The element of Π_t containing state $\omega \in \Omega$ is denoted $\Pi_t(\omega)$. Also let H_t denote the event "$r_t =$" and let $L_t = \Omega \setminus H_t$. Consequences are consumption streams: $X = \mathbb{R}_+^{T+1}$.

A *contingent consumption plan* is a collection $(f_t)_{0\le t\le T}$ such that, for each $t = 0, \dots, T$, $f_t : \Omega \to \mathbb{R}_+$ is Π_t-measurable. Each such collection defines an act $f : \Omega \to X$ by letting $f(\omega) = (f_t(\omega))_{0\le t\le T}$. Denote the set of such acts by $\mathcal{F}_A$, where the subscript "A" suggests that these acts are "adapted" to the filtration $\Pi_0, \dots, \Pi_T$. To keep track of wealth given an act $f \in \mathcal{F}_A$, define $w^f = (w_t^f)_{0\le t\le T}$ by $w_0^f(\omega) = w_0$ and, for $t = 1, \dots, T$, $w_t^f(\omega) = [w_{t-1}^f(\omega) - f_{t-1}(\omega)]r_{t-1}(\omega)$. Finally, let $\mathcal{F}_A(w_0)$ denote the subset of $\mathcal{F}_A$ whose elements f satisfy $f_t(\omega) \in [0, w_t^f]$ for all $t = 0, \dots, T$; these correspond to *feasible* consumption plans.

Preferences and Updating

Assume discounted power utility on X: $u(x) = \sum_{t=0}^{T} \delta^t v(x_t)$ with $v(c) = c^{1-\gamma}/(1-\gamma)$. Let the baseline prior p be uniform on Ω, which reflects the distributional assumptions on $(r_t)_{0\le t<T}$. Next, fix $T-1$ adjustment factors $\zeta = (\zeta_t)_{0\le t<T-1}$, where

$$\zeta_t(\omega) = \begin{cases} \varepsilon & \text{if } r_t(\omega) = r_{t+1}(\omega), \\ -\varepsilon & \text{if } r_t(\omega) \neq r_{t+1}(\omega), \end{cases}$$

and $\varepsilon > 0$ is "suitably small." Observe that ζ_t is Π_{t+2}-measurable; furthermore, it can be verified that $\mathrm{E}_p[\zeta_t] = 0$ for all $0 \le t < T-1$, as required by Definition 1. Finally, let the adjustment function be defined by $A(\varphi) = -\sum_{t=0}^{T-2} |\varphi_t|$ for all $\varphi \in \mathbb{R}^{T-1}$.

The following facts are established in Section S.4 of the supplemental material: First, for all $f \in \mathcal{F}_A$,

$$V(f) = \sum_{t=0}^{T} \delta^t \mathrm{E}_p[v \circ f_t] - \sum_{t=0}^{T-2} \left| \mathrm{E}_p\left[\zeta_t \sum_{s=t+2}^{T} \delta^s v \circ f_s \right] \right|. \tag{10}$$

Moreover, the updating rule in Eq. (7) yields, for each τ and $F \in \Pi_\tau$, a collection $(\zeta_{t,F})_{0\le t<T-1}$ such that, at time τ and conditional on F, acts in L_A are ranked according to the functional[28]

$$V_\tau(f|F) = v \circ f_\tau + \sum_{t=\tau+1}^{T} \delta^{t-\tau} \mathrm{E}_p[v \circ f_t|F] - \sum_{t=\tau-1}^{T-2} \left| \mathrm{E}_p\left[\zeta_{t,F} \sum_{s=t+2}^{T} \delta^{s-\tau} v \circ f_s | F \right] \right|. \tag{11}$$

Equation (9) is also simpler here: for all $a : \Omega \to \mathbb{R}$ and all t,

$$\mathrm{E}_p[\zeta_{t,\Pi_\tau(\omega)} a | \Pi_\tau(\omega)] = \sum_{G \in \Pi_{\tau+1} : G \subset \Pi_\tau(\omega)} \mathrm{E}_p[\zeta_{t,G} a | G]. \tag{12}$$

Analysis of Consumption–Savings Choices

The *consistent-planning* algorithm prescribes that, at each time τ and for any possible cell $f \in \Pi_\tau$, the agent choose the level of savings that maximizes her conditional VEU payoff as per Eq. (11), calculated assuming that consumption–savings choices at all subsequent times $t = \tau+1, \dots, T-1$ and cells $G \in \Pi_t$ (with $G \subset F$) are as determined in prior iterations of the procedure.[29]

This is conceptually straightforward. However, naively computing the expectations in Eq. (11) at time τ as just described is both analytically cumbersome and computationally intensive: for each possible consumption level at time τ, it is necessary to explicitly calculate how this choice would influence all subsequent consumption–savings decisions at times $t > \tau$. In other words, at any decision point, the entire continuation subtree following a consumption choice must be taken into account.

With EU preferences, this is avoided by assigning a *continuation value* to the subtree following each consumption choice; the decision faced at any time τ then effectively reduces to a simple, two-period problem. It will now be shown that, by virtue of Eq. (12), a similar recursive approach is also possible with VEU preferences and baseline-prior updating. The main difference is that, together with a (baseline) continuation value, it is also necessary to iteratively construct a *continuation adjustment* corresponding to each adjustment factor ζ_t.

To initialize the recursion, for every $w \ge 0$, let $V_{T+1}(w) = 0$. Now assume that $V_{\tau+1}$ and $\Phi_{\tau+1,t}$ have been defined for $\tau+1 \le T+1$ and $\tau-1 \le t \le T-2$; fix

[28]In the notation of Eq. (7), $V_F(f) = \sum_{t=0}^{\tau-1} \delta^t \mathrm{E}_p[v \circ f_t|F] + \delta^\tau V_\tau(f|F)$; however, when evaluating continuation plans at time τ, only $V_\tau(f|F)$ is relevant.

[29]A simplifying feature of this example is that ties do not arise.

$F \in \Pi_\tau$ and $w \geq 0$, and let $s^*_{\tau,F}(w)$ be the (unique, as it turns out) solution to the problem

$$\max_{s\in[0,w]} v(w-s) + \delta \mathrm{E}_p[V_{\tau+1}(r_\tau s)|F] - \delta \sum_{t=\tau-1}^{T-2} \left|\Phi_{\tau+1,t}(Hs|F\cap H_\tau) + \Phi_{\tau+1,t}(Ls|F\cap L_\tau)\right|, \tag{13}$$

where, as usual, a summation over an empty index set equals zero. As with EU preferences, it turns out that $s^*_{\tau,F}(w) = \alpha_{\tau,F} w$, where $\alpha_{\tau,F}$ does not itself depend upon w.

To complete the inductive step, define the *baseline continuation value*

$$V_\tau(w) = v(w - s^*_{\tau,F}(w)) + \delta \mathrm{E}_p\big[V_{\tau+1}(\tilde{r} s^*_{\tau,F}(w))|F\big]; \tag{14}$$

then define the *continuation adjustments*

$$\Phi_{\tau,t}(w|F) = \begin{cases} \delta\big\{\Phi_{\tau+1,t}(Hs^*_{\tau,F}(w)|F\cap H_\tau) \\ \quad + \Phi_{\tau+1,t}(Ls^*_{\tau,F}(w)|F\cap L_\tau)\big\}, & \tau-1 \leq t \leq T-2, \\ \zeta_{\tau-2,F}(\omega)V_\tau(w), & \text{for any } \omega\in F,\ t=\tau-2 \end{cases} \tag{15}$$

(the cases $t=\tau-1$ and $t=\tau-2$ also require $t\geq 0$). Observe that continuation adjustments use the same state variable w as the continuation value; however, they also depend upon the conditioning event F. This is required to keep track of the realization of adjustment factors.

The (unique) *recursive solution* to the problem is the act $f^* \in \mathcal{F}_A$ for which consumption $f^*_\tau(\omega)$ at time τ in state $\omega \in F \in \Pi_\tau$ equals $(1-\alpha_{\tau,F})w^{f^*}_\tau(\omega)$. Section S.4 (supplemental material) proves that this coincides with the solution obtained by direct application of the consistent-planning algorithm. A key step of the argument uses Eq. (12) to show that $\Phi_{\tau,t}(w^{f^*}_\tau(\omega)|F) = \mathrm{E}_p[\zeta_{t,F}\sum_{s=t+2}^{T}\delta^{s-\tau} v\circ f^*_s|F]$ for $\omega\in F$: that is, as claimed, the functions $\Phi_{\tau,t}$ keep track of adjustments. As a result, the problem in Eq. (13) is analogous to a two-period decision situation: it is not necessary to explicitly trace out the effects of the choice of s at time τ on subsequent decisions, because the relevant payoff information is encoded in the functions defined in Eqs. (14) and (15).

4.6. *Complementary Independence for Other Decision Models*

This section investigates the implications of the complementary independence axiom for four well known families of preferences: the maxmin-expected utility (MEU) model of Gilboa and Schmeidler (1989), the variational preferences model of Maccheroni, Marinacci, and Rustichini (2006), the Choquet

TABLE II

NECESSARY AND SUFFICIENT CONDITIONS FOR COMPLEMENTARY INDEPENDENCE

Model	Representation $I(a)$	Property of Baseline Prior p
MEU	$\min_{q\in C} \mathrm{E}_q[a] C \subset ba_1(\Sigma)$	$\forall q \in C, 2p - q \in C$
Variational	$\min_{q\in ba_1(\Sigma)}(\mathrm{E}_q[a] + c^*(q))$ u unbounded above or below, $x_f \sim f$ and $c^*(q) = \sup_{f\in\mathcal{F}_0}(u(x_f) - \mathrm{E}_q[u\circ f])$	$\forall q \in ba_1(\Sigma)$, $2p - q \in ba_1(\Sigma) \Rightarrow c^*(q) = c^*(2p - q)$, and $2p - q \notin ba_1(\Sigma) \Rightarrow c^*(q) = \infty$
CEU	$\int a\,dv$, $\int \cdot\,dv$ Choquet integral w.r.t. capacity v	$\forall E \in \Sigma, 1 - v(\Omega \setminus E) = 2p(E) - v(E)$
Smooth	$\int_{ba_1(\Sigma)} \phi(\mathrm{E}_q[a])\,d\mu(q)$ μ has finite support	(Only sufficient) $\forall q \in ba_1(\Sigma)$, $2p - q \in ba_1(\Sigma) \Rightarrow \mu(q) = \mu(2p - q)$ and $2p - q \notin ba_1(\Sigma) \Rightarrow \mu(q) = 0$

expected utility (CEU) model of Schmeidler (1989), and the smooth-ambiguity model of Klibanoff, Marinacci, and Mukerji (2005). In the interest of conciseness, the results are presented in tabular form (see Table II); the reader is referred to the original papers for details on the representations and their axiomatizations, and to Section S.2 of the supplemental material for formal statements and proofs.

The second column in Table II indicates the functional $I : u \circ \mathcal{F}_0 \to \mathbb{R}$ that, along with a utility function $u : X \to \mathbb{R}$, represents preferences in each of these models: that is, for all $f, g \in \mathcal{F}_0$, $f \succcurlyeq g$ if and only if $I(u \circ f) \geq I(u \circ g)$.

Notation: $ba_1(\Sigma)$ is the set of probability charges on (Ω, Σ).

The third column in Table II contains the main results of this subsection. Each entry should be interpreted as follows: the model under consideration satisfies complementary independence (Axiom 7) if and only if there exists a probability $p \in ba_1(\Sigma)$ with the properties indicated in the table. For the smooth-ambiguity model, this condition is only sufficient for Axiom 7.[30] It is also important to notice that, for each of these models, under the stated condition, the baseline probability p is fully characterized by preferences: it is the only probability charge such that, for all complementary pairs of acts $(f, \bar{f})$, $f \succcurlyeq \bar{f}$ if and only if $\mathrm{E}_p[u \circ f] \geq \mathrm{E}_p[u \circ \bar{f}]$.

Table II emphasizes the formal analogy among the various conditions for complementary independence (CI). This allows a unitary interpretation of these conditions.

Consider first the MEU, variational, and smooth models. Fix an act f and compute its baseline EU evaluation $\mathrm{E}_p[u \circ f]$. Suppose that a probability charge q provides a more pessimistic evaluation of f, in the sense that

[30]In the setting of Klibanoff, Marinacci, and Mukerji (2005), it is easy to provide a condition on second-order preferences that is equivalent to the property in Table II and hence implies complementary independence.

$E_p[u \circ f] > E_q[u \circ f]$. It is then immediate to verify that $E_{2p-q}[u \circ f] > E_p[u \circ f]$, so the charge $2p - q$ provides a more optimistic evaluation of f. Indeed, $E_{2p-q}[u \circ f]$ exceeds the baseline $E_p[u \circ f]$ precisely by the amount by which the latter exceeds $E_q[u \circ f]$. For CI to hold in the MEU, variational, and smooth models, the probability charges q and $2p - q$ must *receive the same weight in the representation of preferences*, where the precise meaning of "weight" is model-specific.[31] Informally, under CI, the individual must hold a *balanced* view of probabilistic assessments that are equally pessimistic and optimistic relative to the baseline p. Thus, the latter serves as a cognitive "center of symmetry."

In the CEU model, the set function defined by $E \mapsto 1 - v(\Omega \setminus E)$ is usually called the dual of the capacity v. Furthermore, if v is ambiguity-averse in the sense of Schmeidler (1989), its dual is ambiguity-loving. According to Table II, under CI the dual of v is precisely $2p - v$. Again, this suggests that the baseline p acts as a center of symmetry between capacities representing pessimistic and optimistic evaluations.[32]

This property is satisfied, for instance, in several well known specifications of MEU preferences. For finite state spaces, one important example is provided by *mean-standard deviation preferences*, represented by the functional $V(f) = E_p[u \circ f] - \theta\sigma_p(u \circ f)$ (Grant and Kajii (2007)); analogous representations for general state spaces can be obtained by replacing the standard deviation $\sigma_p(\cdot)$ with a different measure of dispersion, such as the Gini mean difference (Yitzhaki (1982)) to ensure monotonicity. For a different, broad class of MEU examples, consider a finite state space Ω, fix a baseline prior p, and let $C = \{q \in \Delta(\Omega) : \|p - q\| \leq \varepsilon\}$, where $\|\cdot\|$ denotes any ℓ_p norm ($p \geq 1$) on $\mathbb{R}^{|\Omega|}$; this suggests a concern for *robustness* to the misspecification of the baseline prior p. Further details may be found in Siniscalchi (2007).

5. DISCUSSION

5.1. *Related Literature*

In the context of choice under risk, Quiggin and Chambers (1998, 2004) analyzed models featuring an exogenously given, objective reference probability p. Under suitable assumptions, a random variable y is evaluated according to the difference between its expectation $E_p(y)$ with respect to p and a "risk index" $\rho(y)$. See also Epstein (1985) and Safra and Segal (1998).

Similar functional forms also appear in the social-choice literature. A classic result owing to Roberts (1980) characterized social-welfare functionals that evaluate a profile $u_1, \ldots, u_I$ of utility imputations according to the form

[31]For the MEU model, p must be the *barycenter* of the set of priors C; for variational preferences, q and $2p - q$ must be equally "costly"; and in the smooth model, q and $2p - q$ must receive the same second-order probability.

[32]I emphasize that ambiguity aversion is *not* required for the characterization in Table II; however, the interpretation in the text may be more transparent for ambiguity-averse preferences.

$\bar{u} - g(u_1 - \bar{u}, \ldots, u_I - \bar{u})$, where $\bar{u} = \frac{1}{I}\sum_i u_i$. Ben-Porath and Gilboa (1994) characterized orderings over income distributions that can be represented in what is essentially a special case of the VEU functional, with the uniform distribution as reference probability. These contributions suggest an alternative formulation of the VEU representation.

Assume for simplicity that the state space Ω is finite and write $\Omega = \{\omega_0, \ldots, \omega_{n-1}\}$. Also consider a strictly positive probability p on Ω and a utility function u. For every $0 \leq i < n$, let

$$\zeta_i^c(\omega_i) = \frac{1 - p(\{\omega_i\})}{p(\{\omega_i\})} \quad \text{and} \quad \zeta_i^c(\omega_j) = -1 \quad \forall j \neq i. \tag{16}$$

Then, for every $f \in \mathcal{F}_0$, $\mathrm{E}_p[\zeta_i^c \cdot u \circ f] = u(f(\omega_i)) - \mathrm{E}_p[u \circ f]$, so $(\mathrm{E}_p[\zeta_i^c \cdot u \circ f])_{0 \leq i < n}$ is the vector of *statewise utility deviations* from the baseline EU evaluation of f. The VEU representation (u, p, n, ζ^c, A) then takes the *canonical*[33] form $V(f) = \mathrm{E}_p[u \circ f] + A((u(f(\omega_i)) - \mathrm{E}_p[u \circ f])_{0 \leq i < n})$.

The canonical VEU representation is unique and emphasizes the dependence upon the outcomes delivered by an act in every states. Furthermore, it highlights the relationship with the social-choice literature. However, canonical representations are *not sharp*; therefore, it is not possible to identify canonical adjustment factors ζ_i^c with distinct sources of ambiguity.

The literature on model uncertainty, initiated by Hansen, Sargent, and coauthors (see, e.g., Hansen and Sargent (2001), Hansen, Sargent, and Tallarini (1999)), also prominently features a reference prior; the focus in this literature is largely on applications to macroeconomics and finance, rather than on behavioral foundations. An interesting axiomatization has recently been provided by Strzalecki (2007); see also Wang (2003).

A recent paper by Grant and Polak (2007) provided a "primal representation" of Maccheroni, Marinacci, and Rustichini's (2006) variational preferences model in a finite-states setting and generalized it by relaxing translation invariance (monotonicity and ambiguity aversion are also weakened). The representation Grant and Polak proposed is related to the ones in Quiggin and Chambers (2004) and Roberts (1980): each act f is evaluated by aggregating a "reference expected utility" term $\mathrm{E}_p[u \circ f]$, where p is a suitable probability, and an "ambiguity index" $\rho(\cdot)$ that depends upon the statewise utility deviations $u(f(\omega_i)) - \mathrm{E}_p[u \circ f]$. These authors show that, for variational preferences, the aggregator is additive; relaxing translation invariance leads to more general aggregators.

The reference prior p in Grant and Polak (2007) is not unique in general. In the space of utility profiles, p corresponds to a hyperplane supporting the individual's indifference curves at a point on the certainty line. Decision models featuring a kink at certainty (e.g., MEU, CEU, or invariant biseparable

[33] I thank a referee for drawing attention to this particular representation, and suggesting the term "canonical."

preferences) allow for multiple supporting hyperplanes and hence, typically, multiple reference priors. One way to ensure uniqueness is to assume that indifference curves are "flat" or smooth at certainty, but, in this case, the prior p only reflects (indeed, under smoothness, approximates) local behavior around the certainty line. The baseline prior in the VEU representation is instead uniquely identified by preferences over complementary acts. Hence, *every* act contributes to the behavioral identification of the baseline prior.

Furthermore, Grant and Polak maintain a form of ambiguity aversion, which is required for the existence of a supporting hyperplane at certainty; the VEU representation instead allows for arbitrary ambiguity attitudes. Finally, the ambiguity index ρ in Grant and Polak (2007) is not invariant to sign changes; the VEU adjustment functional A instead satisfies this invariance property, which supports the intuition that adjustments to baseline evaluations reflect outcome variability, or dispersion. On the other hand, the analysis of VEU preferences provided in this paper does assume and rely upon translation invariance (cf. Axiom 5); however, see Section 5.2 below.

Decision models that incorporate a reference prior have also been analyzed in environments where the objects of choice either consist of or include sets of probabilities. In Stinchcombe (2003), Gajdos, Tallon, and Vergnaud (2004b), and Gajdos, Hayashi, Tallon, and Vergnaud (2008), the reference prior is characterized as the Steiner point of the set of probabilities under consideration. In Gajdos, Tallon, and Vergnaud (2004a) and Wang (2003), each object of choice explicitly indicates the reference prior. The present paper complements the analysis of these authors by characterizing a decision model that features a baseline prior in a fully subjective environment.

Kopylov (2006) axiomatized a special case of MEU preferences, where the characterizing set of priors is generated by ε-contamination: that is, it takes the form $\{(1-\varepsilon)p+\varepsilon q : q \in \Delta\}$, where p serves as a reference prior and Δ is a set of "contaminating" probability measures. While the prior p is endogenously derived, the set Δ must be specified exogenously. Chateauneuf, Eichberger, and Grant (2007) characterized CEU with respect to a "neo-additive" capacity; this model can be viewed as α-maxmin expected utility with a set of priors obtained by ε-contamination, in which the reference prior and the "contaminating set" are both endogenously derived.

Finally, as was noted following Corollary 2 and elsewhere, VEU preferences that satisfy Schmeidler's ambiguity-aversion assumption (i.e., Axiom 9) are also variational preferences. In this case, the VEU representation can provide a convenient alternative to the variational specification. To elaborate, recall that in the canonical variational representation (cf. the second row in Table II), the utility index $V(f)$ assigned to an act f is the value of a minimization problem: $V(f) = \min_{q \in ba_1(\Sigma)} \mathrm{E}_q[u \circ f] + c^*(q)$. In general, there may be no closed-form solution to this problem and hence no explicit expression for the utility index $V(f)$.[34] On the other hand, the VEU utility index $V(f)$ is explicitly defined in Eq. (3); VEU representations with a concave function A can thus provide a family of richly parameterized, analytically convenient specifications of variational preferences. Furthermore, Theorem 1 and Corollary 2 provide a full behavioral characterization of preferences that are both VEU and variational. It is worth emphasizing, however, that VEU preferences enable the modeler to capture more nuanced forms of aversion to ambiguity than are allowed by maxmin EU or variational preferences (cf. Section 4.3).

5.2. *Additional Features and Extensions*

Probabilistic Sophistication

Non-EU VEU preferences can be probabilistically sophisticated in the sense of Machina and Schmeidler (1992). A characterization of probabilistic sophistication for VEU preferences is left for future work; Section S.3 in the supplemental material provides a simple, related result that sheds further light on the central role of baseline probabilities in the VEU model. Given a preference relation $\succcurlyeq$ on $\mathcal{F}_0$, define the induced *likelihood ordering* $\succcurlyeq_\ell \subset \Sigma \times \Sigma$ by

$$\forall E, F \in \Sigma, \quad E \succcurlyeq_\ell F \quad \Leftrightarrow \quad xEy \succcurlyeq xFy \quad \text{for all } x, y \in X \text{ with } x \succ y.$$

Proposition 10 (supplemental material) shows that the likelihood ordering induced by a VEU preference is represented by a probability measure μ if and only if μ is its baseline prior.

Translation-Invariance

Because they satisfy the weak certainty independence axiom (Axiom 5), VEU preferences are invariant to "translation in utility space"; in the language of Grant and Polak (2007), they display "constant absolute ambiguity aversion," as do, for instance, MEU, CEU, variational, and invariant-biseparable preferences. However, this is *solely* a consequence of Axiom 5: the key axiom in the characterization of the VEU representation, namely complementary independence (Axiom 7), does not imply or require translation-invariance.

For instance, consider the smooth-ambiguity model of Klibanoff, Marinacci, and Mukerji (2005): Section 4.6 provides a sufficient condition for complementary independence that involves only the second-order probability μ, but *not* the second-order utility ϕ; the latter is *unrestricted*. Smooth-ambiguity preferences are translation-invariant if and only if ϕ is negative exponential or linear; it then follows that there exists a rich class of smooth-ambiguity preferences

[34] Hansen and Sargent's (2001) multiplier preferences are variational preferences for which the minimization problem *does* have a closed-form solution; their popularity in applications is probably due in part to this fact.

that are not translation-invariant, but nevertheless satisfy complementary independence.

For a different perspective on this issue, consider an "aggregator" function $W:\mathbb{R}^2 \to \mathbb{R}$, strictly increasing in both arguments. Also let u, p, ζ, and A be as in the VEU representation. Then one may consider preferences defined by letting, for all $f, g \in \mathcal{F}_0$,

$$f \succcurlyeq g \quad \Leftrightarrow \quad W\big(\mathrm{E}_p[u \circ f], A(\mathrm{E}_p[\zeta \cdot u \circ f])\big) \\ \geq W\big(\mathrm{E}_p[u \circ g], A(\mathrm{E}_p[\zeta \cdot u \circ g])\big).$$

The representation in this paper corresponds to the aggregator $W(x, y) = x + y$. It is then easy to verify that Axiom 7 holds for such preferences, even if they are not translation-invariant.

Therefore, it may be possible to characterize a version of the VEU representation that does not impose "constant absolute ambiguity aversion." The resulting model would still feature sign- and translation-invariant *adjustments* $A(\mathrm{E}_p[\zeta \cdot u \circ f])$, and hence would be consistent with the variability interpretation described in this paper.[35] Such an extension is left to future work.

APPENDIX A: CONDITIONS FOR MONOTONICITY

REMARK 2: *If a tuple* (u, p, n, ζ, A) *satisfies parts* 1 *and* 2 *in Definition* 1, $n < \infty$, *and* A *is continuous on* $\mathcal{E}(u \circ \mathcal{F}_0; p, \zeta)$ *and differentiable on* $\mathcal{E}(u \circ \mathcal{F}_0; p, \zeta) \setminus A^{-1}(0)$, *then it satisfies part* 3 *if and only if* $p(E) + \sum_{0 \leq i < n} (\partial A/\partial \varphi_i)(\varphi)\mathrm{E}_p[\zeta_1 1_E] \geq 0$ *for all* $\varphi \notin A^{-1}(0)$ *and* $E \in \Sigma$.

PROOF: Part 3 is easily seen to be equivalent to the following condition: for all $a \in B_0(\Sigma, u(X))$, $E \in \Sigma$, and $\varepsilon > 0$ such that $a + \varepsilon 1_E \in B_0(\Sigma, u(X))$,

$$\varepsilon p(E) + A(\mathrm{E}_p[\zeta \cdot a] + \varepsilon \mathrm{E}_p[\zeta \cdot 1_E]) - A(\mathrm{E}_p[\zeta \cdot a]) \geq 0. \tag{17}$$

For any $\varphi \in \mathcal{E}(u \circ \mathcal{F}_0; p, \zeta)$, if $A(\varphi) = 0$ or $\varphi = \mathrm{E}_p[\zeta \cdot a]$ and $a + 1_E \varepsilon \in u \circ \mathcal{F}_0$ for some $\varepsilon > 0$, Eq. (17) readily implies the condition in the remark; if $A(\varphi) \neq 0$, $\varphi = \mathrm{E}_p[\zeta \cdot a]$, but $a + 1_E \varepsilon \notin u \circ \mathcal{F}_0$ for any $\varepsilon > 0$, then let $F = \{\omega : a(\omega) = \max u(X)\}$; since a is a simple function, $F \neq \emptyset$. Consider the sequence (a_k) given by $a_k = a - 1_F \frac{1}{k}$; for k large, $a_k \in u \circ \mathcal{F}_0$, $A(\mathrm{E}_p[\zeta \cdot a_k]) \neq 0$, and there is $\varepsilon_k > 0$ such that $a_k + 1_E \varepsilon_k \in u \circ \mathcal{F}_0$. Then $p(E) + \sum_{0 \leq i < n} (\partial A/\partial \varphi_i)(\mathrm{E}_p[\zeta \cdot a_k])\mathrm{E}_p[\zeta_i \cdot 1_E] \geq 0$ for all large k, and the claim follows by continuity of the partial derivatives $\partial A/\partial \varphi_i$.

Now suppose the condition in the remark holds and fix $a, E, \varepsilon > 0$ such that $a, a + 1_E \varepsilon \in u \circ \mathcal{F}_0$. To simplify the notation, write $\varphi_\eta = \mathrm{E}_p[\zeta \cdot a] + \eta \mathrm{E}_p[\zeta \cdot 1_E]$ for all $\eta \in [0, \varepsilon]$.

[35] Axiom 8 would also have to be dropped: after all, its interpretation involves translation-invariance. In any case, recall that its role is limited even in the present setting.

Consider first the case $A(\varphi_0) = 0$. Let $\varepsilon_0 = \sup\{\eta \in [0, \varepsilon] : A(\varphi_\eta) = 0\}$. If $\varepsilon_0 = 0$, then $A(\varphi_\eta)$ is differentiable for all $\eta \in (0, \varepsilon)$ and

$$(18) \qquad \begin{aligned} &\varepsilon p(E) + A(\varphi_\varepsilon) - A(\varphi_0) \\ &\quad = 0 \cdot p(E) + A(\varphi_0) - A(\varphi_0) \\ &\qquad + \int_0^\varepsilon \left[p(E) + \sum_{0 \le i < n} \frac{\partial}{\partial \varphi_i} A(\varphi_\eta) \mathrm{E}_p[\zeta_i \cdot 1_E] \right] d\eta \\ &\quad \ge 0, \end{aligned}$$

as required. If $\varepsilon_0 > 0$, then by continuity $A(\varphi_{\varepsilon_0}) = 0 = A(\varphi_0)$, so

$$(19) \qquad \varepsilon_0 p(E) + A(\varphi_{\varepsilon_0}) - A(\varphi_0) = \varepsilon_0 p(E) \ge 0.$$

Thus, in particular, if $\varepsilon_0 = \varepsilon$, Eq. (17) holds. If instead $\varepsilon_0 < 1$, then one can repeat the preceding argument with $a' = a + \varepsilon_0 1_E$ and $\varepsilon' = \varepsilon - \varepsilon_0$ in lieu of a and ε. By assumption $A(\mathrm{E}_p[\zeta \cdot a'] + \eta \mathrm{E}_p[\zeta \cdot 1_E]) \neq 0$ for all $\eta \in (0, \varepsilon')$, so the argument just given implies that $(\varepsilon - \varepsilon_0) p(E) + A(\varphi_\varepsilon) - A(\varphi_{\varepsilon_0}) \ge 0$. Together with Eq. (19), this implies that Eq. (17) holds in this case as well.

Consider now the case $A(\varphi_0) > 0$. Let $\varepsilon_1 = \sup\{\eta \in [0, \varepsilon] : A(\varphi_\eta) \neq 0\}$. By continuity of A, $\varepsilon_1 > 0$; thus, integrating on $(0, \varepsilon_1)$ as in Eq. (18) yields $\varepsilon_1 p(E) + A(\varphi_{\varepsilon_1}) - A(\varphi_0) \ge 0$. If $\varepsilon_1 = \varepsilon$, the proof is complete; otherwise, note that by continuity of A, $A(\varphi_{\varepsilon_1}) = 0$. Applying the argument given above to $a' = a + \varepsilon_1 1_E$ and $\varepsilon' = \varepsilon - \varepsilon_1$ in lieu of a and ε yields $(\varepsilon - \varepsilon_1) p(E) + A(\varphi_\varepsilon) - A(\varphi_{\varepsilon'}) \ge 0$; together with $\varepsilon_1 p(E) + A(\varphi_{\varepsilon_1}) - A(\varphi_0) \ge 0$, this implies that Eq. (17) holds. *Q.E.D.*

REMARK 3: *If* (u, p, n, ζ, A) *satisfies parts* 1 *and* 2 *in Definition* 1, *and* A *is concave and positively homogeneous*, *then* (u, p, n, ζ, A) *satisfies part* 3 *if and only if* $p(E) + A(\mathrm{E}_p[\zeta \cdot 1_E]) \ge 0 \ \forall E \in \Sigma$.

PROOF: Since A is positively homogeneous, it has a unique positively homogeneous extension to $\mathcal{E}(B_0(\Sigma); p, \zeta)$ given by $A(\mathrm{E}_p[\zeta \cdot \alpha a]) = \alpha A(\mathrm{E}_p[\zeta \cdot a])$ for all $\alpha > 0$ and $a \in u \circ \mathcal{F}_0$. Hence, $A(\mathrm{E}_p[\zeta \cdot a])$ is well defined for all $a \in B_0(\Sigma)$, and A is concave on this domain. Hence, for all $\varphi, \psi \in \mathcal{E}(B_0(\Sigma); p, \zeta)$, $A(\varphi) = A(\psi + (\varphi - \psi)) = 2A(\frac{1}{2}\psi + \frac{1}{2}(\varphi - \psi)) \ge 2\frac{1}{2}A(\psi) + 2\frac{1}{2}A(\varphi - \psi)$, so $A(\varphi - \psi) \le A(\varphi) - A(\psi)$.

Now suppose that $p(E) + A(\mathrm{E}_p[\zeta \cdot 1_E]) \ge 0$ for all $E \in \Sigma$ and consider $a, b \in B_0(\Sigma)$ with $a(\omega) \ge b(\omega)$ for all ω. Then $a - b \in B_0(\Sigma)$, and since $a(\omega) - b(\omega) \ge 0$ for all ω, concavity and homogeneity, together with linearity and monotonicity of $\int \cdot \, dp$, imply that $\int (a - b) \, dp + A(\mathrm{E}_p[\zeta \cdot (a - b)]) \ge 0$. But the argument given above implies that $A(\mathrm{E}_p[\zeta \cdot (a - b)]) \le A(\mathrm{E}_p[\zeta \cdot a]) - A(\mathrm{E}_p[\zeta \cdot b])$, so $\int a \, dp + A(\mathrm{E}_p[\zeta \cdot a]) \ge \int b \, dp + A(\mathrm{E}_p[\zeta \cdot b])$. The other direction is immediate. *Q.E.D.*

APPENDIX B: PROOFS

B.1. *Additional Notation and Preliminaries on Niveloids*

The indicator function of an event $E \in \Sigma$ will be denoted by 1_E. Inequalities between two elements a, b of $B(\Sigma)$ are interpreted pointwise: $a \geq b$ means that $a(\omega) \geq b(\omega)$ for all $\omega \in \Omega$.

Let $\Phi \subset B(\Sigma)$ be convex. A functional $I : \Phi \to \mathbb{R}$ is a *niveloid* iff $I(a) - I(b) \leq \sup(a - b)$ for all $a, b \in \Phi$; it is *normalized* if $I(\gamma 1_\Omega) = \gamma$ for all $\gamma \in \mathbb{R}$ such that $\gamma 1_\Omega \in \Phi$; it is *monotonic* iff, for all $a, b \in \Phi$, $a \geq b$ implies $I(a) \geq I(b)$; it is *constant-mixture invariant* iff, for all $a \in \Phi$, $\alpha \in (0, 1)$, and $\gamma \in \mathbb{R}$ with $\gamma 1_\Omega \in \Phi$, $I(\alpha a + (1 - \alpha)\gamma) = I(\alpha a) + (1 - \alpha)\gamma$; it is *vertically invariant* iff $I(a + \gamma) = I(a) + \gamma$ for all $a \in \Phi$ and $\gamma \in \mathbb{R}$ such that $a + \gamma \in \Phi$; and it is *affine* iff, for all $a, b \in \Phi$ and $\alpha \in (0, 1)$, $I(\alpha a + (1 - \alpha)b) = \alpha I(a) + (1 - \alpha)I(b)$. Maccheroni, Marinacci, and Rustichini (2006) (MMR henceforth) demonstrated the usefulness of niveloids in decision theory and established useful results reviewed below.

If $\Phi = B_0(\Sigma)$ or $\Phi = B(\Sigma)$, then a functional $I : \Phi \to \mathbb{R}$ is *positively homogeneous* iff, for all $a \in \Phi$ and $\alpha \geq 0$, $I(\alpha a) = \alpha I(a)$; is *c-additive* iff $I(a + \alpha) = I(a) + \alpha$ for all $\alpha \in \mathbb{R}_+$ and $a \in \Phi$; is *additive* iff $I(a + b) = I(a) + I(b)$ for all $a, b \in \Phi$; is *c-linear* iff it is c-additive and positively homogeneous; and is *linear* iff it is additive and positively homogeneous.

Let $ba(\Sigma)$ and $ba_1(\Sigma)$ denote, respectively, the set of finitely additive measures and the set of charges (finitely additive probabilities) on (Ω, Σ). Recall that $ba(\Sigma)$ is isometrically isomorphic to the norm dual of $B_0(\Sigma)$ and $B(\Sigma)$. Also, the $\sigma(ba(\Sigma), B(\Sigma))$ and $\sigma(ba(\Sigma), B_0(\Sigma))$ topologies coincide on $ba_1(\Sigma)$; they are referred to as the weak* topology.

Furthermore, if $\Gamma \subset \mathbb{R}$ is a nonempty, nonsingleton interval, denote by $B_0(\Sigma, \Gamma)$ and $B(\Sigma, \Gamma)$ the restrictions of $B_0(\Sigma)$ and $B(\Sigma)$ to functions taking values in Γ. Then the weak* topology on $ba_1(\Sigma)$ also coincides with the $\sigma(ba(\Sigma), B_0(\Sigma, \Gamma))$ and $\sigma(ba(\Sigma), B(\Sigma, \Gamma))$ topologies.

The following useful results on niveloids are owing to or reviewed in MMR. In particular, item 6 provides a first representation for preferences satisfying Axioms 1–5.

PROPOSITION 6—MMR: *Let Γ be an interval such that $0 \in \operatorname{int}(\Gamma)$ and $I : B_0(\Sigma, \Gamma) \to \mathbb{R}$.*

1. *If I is a niveloid, it is supnorm, hence Lipschitz continuous.*
2. *If $I : B_0(\Sigma, K) \to \mathbb{R}$ is a niveloid, then it has a (minimal) niveloidal extension to $B(\Sigma)$.*
3. *I is a niveloid iff it is monotonic and constant-mixture invariant.*
4. *If I is constant-mixture invariant, then it is vertically invariant.*
5. *If I is vertically invariant, then it has a unique, vertically invariant extension $\hat{I}$ to $B_0(\Sigma, \Gamma) + \mathbb{R} \equiv \{a + 1_\Omega \gamma : a \in B_0(\Sigma, \Gamma), \gamma \in \Gamma\}$.*

6. $\succcurlyeq$ *on* $\mathcal{F}_0$ *satisfies Axioms* 1–5 *if and only if there is a nonconstant, affine function* $u: X \to \mathbb{R}$ *and a normalized niveloid* $I: B_0(\Sigma, u(X)) \to \mathbb{R}$ *such that* $f \succcurlyeq g$ *iff* $I(u \circ f) \geq I(u \circ g)$.

The following uniqueness and extension results are straightforward and useful:

COROLLARY 3: *If* I, u *and* I', u' *provide two representations of* $\succcurlyeq$ *as per the last point of Proposition* 6, *then* $u' = \alpha u + \beta$ (*with* $\alpha > 0$) *and* $I'(\alpha a + \beta) = \alpha I(a) + \beta$ *for all* $a \in B_0(\Sigma, u(X))$.

PROOF: Since I and I' are normalized, standard results imply that $u' = \alpha u + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Next, for every $a \in B_0(\Sigma, \Gamma)$, let $f \in \mathcal{F}_0$ be such that $u \circ f = a$ and $x \sim f$: thus, since I and I' are normalized, $u(x) = I(u \circ f) = I(a)$ and similarly $u'(x) = I'(u' \circ f)$; that is, $\alpha u(x) + \beta = I'(\alpha u \circ f + \beta)$ and therefore $\alpha I(a) + \beta = I'(\alpha a + \beta)$. [Note that this is consistent with normalization: $\alpha I(\gamma 1_\Omega) = \alpha\gamma$ and $I'(\alpha\gamma 1_\Omega) = \alpha\gamma$.] *Q.E.D.*

COROLLARY 4: *A niveloid* $I: B_0(\Sigma, \Gamma) \to \mathbb{R}$ *admits a unique niveloidal extension to* $B(\Sigma, \Gamma)$. *Therefore, if a preference* $\succcurlyeq$ *on* $\mathcal{F}_0$ *admits a niveloidal representation* (I, u) *as in part* 6 *of Proposition* 6, *then it admits a unique extension to* $\mathcal{F}_b$ *that satisfies Axioms* 1–5. *Together with* u, *the extension of* I *to* $B(\Sigma, \Gamma)$ *represents the extension of* $\succcurlyeq$ *to* $\mathcal{F}_b$.

PROOF: By Proposition 6, there is a minimal niveloidal extension of I to $B(\Sigma)$; let $\hat{I}$ be its restriction to $B(\Sigma, \Gamma)$. If there is another niveloidal extension $\hat{I}'$ of I to $B(\Sigma, \Gamma)$, fix $a \in B(\Sigma, \Gamma)$ and a sequence $a^k \to a$ such that $a^k \in B_0(\Sigma, \Gamma)$ for all k. Then $\hat{I}(a) = \lim_k \hat{I}(a^k) = \lim_k I(a^k) = \lim_k \hat{I}'(a^k) = \hat{I}'(a)$.

Now define $\hat{\succcurlyeq}$ on $\mathcal{F}_b$ by $f \hat{\succcurlyeq} g$ iff $\hat{I}(u \circ f) \geq \hat{I}(u \circ g)$ for all $f, g \in \mathcal{F}_b$. One can verify that this defines a preference relation that satisfies Axioms 1–5. Moreover, consider a preference $\hat{\succcurlyeq}'$ that satisfies the same axioms and coincides with $\succcurlyeq$ to $\mathcal{F}_b$. The proof of Lemma 28 in MMR applies verbatim to a preference defined on $\mathcal{F}_b$ and yields a representation $(\hat{I}', u')$, where $\hat{I}'$ is a niveloid defined on $u' \circ \mathcal{F}_b$. Since $\mathcal{F}_0 \subset \mathcal{F}_b$, we can take $u' = u$ and $\hat{I}' = I$ on $u \circ \mathcal{F}_0$. But then $\hat{I}' = \hat{I}$, which implies that $\hat{\succcurlyeq}' = \hat{\succcurlyeq}$. *Q.E.D.*

NOTE: For notational simplicity, the unique extension of a niveloid $I: B_0(\Sigma, \Gamma)$ to $B(\Sigma, \Gamma)$ will also be denoted by I.

B.2. *Characterization of Complementary Independence and Crisp Acts*

This subsection starts with the "niveloidal representation" of $\succcurlyeq$ provided by part 6. It will first be shown that Axioms 8 and 7 hold if and only if a "baseline linear functional" J can be defined. This identifies a baseline prior. Then, it will be shown that I coincides with J on all crisp acts. Finally, further properties of the set of crisp acts are investigated.

To simplify the exposition, throughout this section we maintain the following assumption and definitions: $\succcurlyeq$ is represented by I, u as in Proposition 6, with $0 \in \operatorname{int}(u(X))$. The unique extension of I to $B(\Sigma, u(X))$, and hence to $u \circ \mathcal{F}_b$, is implicitly used wherever it is needed.

Define $J : u \circ \mathcal{F}_b \to \mathbb{R}$ by letting, for all $a \in u \circ \mathcal{F}_b$ and $\gamma \in \mathbb{R}$ with $\gamma - a \in u \circ \mathcal{F}_b$,

$$J(a) = \frac{1}{2}\gamma + \frac{1}{2}I(a) - \frac{1}{2}I(\gamma - a). \tag{20}$$

LEMMA 1: *J is a well defined, normalized niveloid. If $\succcurlyeq$ satisfies Axioms 7 and 8 on $\mathcal{F}_0$, then J is affine on $\mathcal{F}_0$ and has a unique, normalized, and positive linear extension to $B(\Sigma)$, also denoted J. Conversely, if J is affine on $u \circ \mathcal{F}_0$ (resp. $u \circ \mathcal{F}_b$), then $\succcurlyeq$ (resp. the extension of $\succcurlyeq$ to $\mathcal{F}_b$) satisfies Axioms 7 and 8.*

PROOF: J as above is well defined. First, for every $a \in u \circ \mathcal{F}_b$, if $\gamma = \inf_\Omega a + \sup_\Omega a$, then $\gamma - a = \sup_\Omega a - [a - \inf_\Omega a] \in u \circ \mathcal{F}_b$. Furthermore, if $\gamma, \gamma' \in \mathbb{R}$ are such that $\gamma - a, \gamma - a' \in u \circ \mathcal{F}_b$, then $\gamma - a = (\gamma' - a) + (\gamma - \gamma')$, so vertical invariance of I implies that $I(\gamma - a) = I(\gamma' - a) + \gamma - \gamma'$, and so $\frac{1}{2}\gamma - \frac{1}{2}I(\gamma - a) = \frac{1}{2}\gamma - \frac{1}{2}I(\gamma' - a) - \frac{1}{2}(\gamma - \gamma') = \frac{1}{2}\gamma' - \frac{1}{2}I(\gamma' - a)$, as required. Next, J is normalized: if $\gamma \in u(X)$, then $\gamma - \gamma = 0 \in u(X)$, so $J(\gamma) = \frac{1}{2}\gamma + \frac{1}{2}I(\gamma) - \frac{1}{2}I(\gamma - \gamma) = \frac{1}{2}\gamma + \frac{1}{2}\gamma + 0 = \gamma$, because I is normalized and $0 \cdot 1_\Omega \in u \circ \mathcal{F}_b$. Finally, J is a niveloid: for $a, b \in u \circ \mathcal{F}_b$, if $\alpha, \beta \in u(X)$ are such that $\alpha - a, \beta - b \in u \circ \mathcal{F}_b$, then

$$\begin{aligned}
2[J(a) - J(b)] &= \alpha + I(a) - I(\alpha - a) - \beta - I(b) + I(\beta - b) \\
&\le (\alpha - \beta) + \sup_\Omega(a - b) + \sup_\Omega(\beta - b - \alpha + a) \\
&= 2\sup_\Omega(a - b).
\end{aligned}$$

Turn now to Axioms 8 and 7.

First, it will be shown that $\succcurlyeq$ satisfies Axiom 8 if and only if $J(\frac{1}{2}a) = \frac{1}{2}J(a)$ for all $a \in u \circ \mathcal{F}$. Specifically, let $\mathcal{F}$ denote either $\mathcal{F}_0$ or $\mathcal{F}_b$. Fix $f, \bar{f}, x$, and $\bar{x}$ as in Axiom 8 and let $a \in u \circ \mathcal{F}$ and $\gamma \in \mathbb{R}$ be such that $a = u \circ f$ and $\gamma - a = u \circ \bar{f}$. Then $\frac{1}{2}f + \frac{1}{2}\bar{x} \sim \frac{1}{2}\bar{f} + \frac{1}{2}x$ iff $I(\frac{1}{2}a + \frac{1}{2}u(\bar{x})) = I(\frac{1}{2}\bar{f} + \frac{1}{2}u(x))$; by vertical invariance [note that $\frac{1}{2}a, \frac{1}{2}(\gamma - a) \in u \circ \mathcal{F}$] and the properties of $x, \bar{x}$, this equals

$$I\left(\frac{1}{2}a\right) + \frac{1}{2}I(\gamma - a) = I\left(\frac{1}{2}(\gamma - a)\right) + \frac{1}{2}I(a).$$

By the definition of J, rearranging terms, this holds iff $J(\frac{1}{2}a) - \frac{1}{4}\gamma = \frac{1}{2}[J(a) - \frac{1}{2}\gamma]$, that is, $J(\frac{1}{2}a) = \frac{1}{2}J(a)$. Thus, if J has this property, then Axiom 8 holds. Conversely, for any $a \in u \circ \mathcal{F}$, there is $f \in \mathcal{F}$ such that $u \circ f = a$, and as noted in the first part of this proof, one can find $\gamma \in \mathbb{R}$ with $\gamma - a \in u \circ \mathcal{F}$. Again, there will be $\bar{f} \in \mathcal{F}$ with $u \circ \bar{f} = \gamma - a$, so that $f, \bar{f}$ are complementary: if Axiom 8 holds, the argument just given shows that $J(\frac{1}{2}a) = \frac{1}{2}J(a)$.

Now assume that J is affine on $\mathcal{F}_b$. Then, in particular, for all $a \in \mathcal{F}_b$, $J(\frac{1}{2}a) = J(\frac{1}{2}a + \frac{1}{2}\cdot 0) = \frac{1}{2}J(a) + \frac{1}{2}J(0) = \frac{1}{2}J(a)$, and, as shown above, in this case Axiom 8 holds. Next, consider $(f, \bar{f})$, $(g, \bar{g})$, and α as in Axiom 7. Let $a = u \circ f$ and $b = u \circ g$, and let $z, z' \in \mathbb{R}$ be such that $\frac{1}{2}u(f(\omega)) + \frac{1}{2}u(\bar{f}(\omega)) = z$, $\frac{1}{2}u(g(\omega)) + \frac{1}{2}u(\bar{g}(\omega)) = z'$ for all ω. Finally, let $\bar{a} = 2z - a$ and $\bar{b} = 2z' - b$, so $\bar{a} = u \circ \bar{f}$ and $\bar{b} = u \circ \bar{g}$. Then $f \succcurlyeq \bar{f}$ and $g \succcurlyeq \bar{g}$ imply $I(a) \geq I(\bar{a}) = I(2z - a)$, so $J(a) = z + \frac{1}{2}I(a) - \frac{1}{2}I(2z - a) \geq z$; similarly, $J(b) \geq z'$. If J is affine, then $J(\alpha a + (1-\alpha)b) = \alpha J(a) + (1-\alpha)J(b) \geq [\alpha z + (1-\alpha)z']$, so

$$\begin{aligned}
&I(\alpha a + (1-\alpha)b) - I(\alpha\bar{a} + (1-\alpha)\bar{b}) \\
&\quad = I(\alpha a + (1-\alpha)b) - I(\alpha[2z - a] + (1-\alpha)[2z' - b]) \\
&\quad = I(\alpha a + (1-\alpha)b) - I\big(2[\alpha z + (1-\alpha)z'] - \alpha a - (1-\alpha)b\big) \\
&\quad = 2J(\alpha a + (1-\alpha)b) - 2[\alpha z + (1-\alpha)z'] \geq 0,
\end{aligned}$$

where the last equality follows from the definition of J. Thus, $\alpha f + (1-\alpha)g \succcurlyeq \alpha\bar{f} + (1-\alpha)\bar{g}$, that is, Axiom 7 holds.

Conversely, assume that Axioms 8 and 7 hold on $\mathcal{F}_0$. As shown above, $J(\frac{1}{2}a) = \frac{1}{2}J(a)$ for all $a \in u \circ \mathcal{F}_0$. It will now be shown that $J(\frac{1}{2}a + \frac{1}{2}b) = \frac{1}{2}J(a) + \frac{1}{2}J(b)$ for all $a, b \in u \circ \mathcal{F}_0$.

Since $0 \in \operatorname{int}(u(X))$, there is $\delta > 0$ such that $[-\delta, \delta] \subset u(X)$. Assume first that $\|a\|, \|b\| \leq \frac{1}{2}\delta$; this implies that (a) $a, b, -a, -b \in B_0(\Sigma, u(X))$ and, furthermore, (b) $a - J(a), b - J(b), J(a) - a, J(b) - b \in B_0(\Sigma, u(X))$, because monotonicity of J implies that $J(a), J(b) \in [-\frac{1}{2}\delta, \frac{1}{2}\delta]$. Let $f, g, \bar{f}, \bar{g} \in \mathcal{F}_0$ be such that $a - J(a) = u \circ f$, $b - J(b) = u \circ g$, $J(a) - a = u \circ \bar{f}$, and $J(b) - b = u \circ \bar{g}$. Clearly, $(f, \bar{f})$ and $(g, \bar{g})$ are complementary pairs. Furthermore, applying the definition of J with $\gamma = 0$, $J(a - J(a)) = \frac{1}{2}I(a - J(a)) - \frac{1}{2}I(J(a) - a)$ and similarly $J(b - J(b)) = \frac{1}{2}I(b - J(b)) - \frac{1}{2}I(J(b) - b)$. Finally, by vertical invariance of J, $J(a - J(a)) = J(a) - J(a) = 0$ and similarly $J(b - J(b)) = 0$. Thus, $f \sim \bar{f}$ and $g \sim \bar{g}$, so Axiom 7 implies that $\frac{1}{2}f + \frac{1}{2}g \sim \frac{1}{2}\bar{f} + \frac{1}{2}\bar{g}$. It follows that $I(\frac{1}{2}[a - J(a)] + \frac{1}{2}[b - J(b)]) = I(\frac{1}{2}[J(a) - a] + \frac{1}{2}[J(b) - b])$ or $J(\frac{1}{2}[a - J(a)] + \frac{1}{2}[b - J(b)]) = 0$, but by vertical invariance of J, this is equivalent to $J(\frac{1}{2}a + \frac{1}{2}b) = \frac{1}{2}J(a) + \frac{1}{2}J(b)$, as claimed.

Now, for arbitrary $a, b \in B_0(\Sigma, u(X))$, there is an integer $K > 0$ such that $2^{-K}\|a\|, 2^{-K}\|b\| \leq \frac{1}{2}\delta$. Then the argument just given shows that $J(\frac{1}{2}(2^{-K}a) +$

$\frac{1}{2}(2^{-K})b) = \frac{1}{2}J(2^{-K}a) + \frac{1}{2}J(2^{-K}b)$, but it was shown above that, for all $c \in B_0(\Sigma, u(X))$, $J(\frac{1}{2}c) = \frac{1}{2}J(c)$, so it follows that

$$J\left(\frac{1}{2}a + \frac{1}{2}b\right) = 2^K J\left(2^{-K}\left(\frac{1}{2}a + \frac{1}{2}b\right)\right)$$
$$= 2^K \frac{1}{2} J(2^{-K}a) + 2^K \frac{1}{2} J(2^{-K}b) = \frac{1}{2}J(a) + \frac{1}{2}J(b).$$

This implies that $J(\alpha a + (1-\alpha)b) = \alpha J(a) + (1-\alpha)J(b)$ for all dyadic rationals $\alpha = k2^{-K}$, with $k \in \{0, \dots, K\}$ for some integer $K > 0$.[36] But since these are dense in $[0, 1]$ and J is sup-norm continuous, J is affine. The extension of J to $B(\Sigma)$ is now standard. *Q.E.D.*

By standard results, if J is linear, there exists a unique $p \in ba_1(\Sigma)$ such that

$$(21) \qquad \forall a \in B(\Sigma), \quad J(a) = \int_\Omega a\, dp.$$

OBSERVATION 1: Note that, if $f, \bar{f}$ are complementary acts, then $f \succcurlyeq \bar{f}$ iff $J(u \circ f) \geq J(u \circ \bar{f})$. Thus, J is identified by preferences over complementary acts. Lemma 1 then shows that if Axioms 7 and 8 hold, such preferences identify the baseline prior p.

To investigate further properties of the functional I, a short detour is needed. Begin by defining and characterizing a binary relation, to be interpreted as "unambiguous preference." The following lemma adapts notions and employs results from Ghirardato, Maccheroni, and Marinacci (2004) (GMM henceforth). Since its proof merely adapts arguments from GMM, it is relegated to the supplemental material.

LEMMA 2: *There exists a unique, weak* compact and convex set $\mathcal{C} \subset ba_1(\Sigma)$ such that, for all $a, b \in B_0(\Sigma, u(X))$,*

$$(22) \qquad \forall \alpha \in (0,1], c \in B_0(\Sigma, u(X)): \quad I(\alpha a + (1-\alpha)c) \geq I(\alpha b + (1-\alpha)c)$$
$$\iff \quad \forall q \in \mathcal{C}: \quad \int a\, dq \geq \int b\, dq.$$

Furthermore, for all $a, b \in B(\Sigma, u(X))$,

$$(23) \qquad \forall \alpha \in (0,1], c \in B(\Sigma, u(X)): \quad I(\alpha a + (1-\alpha)c) \geq I(\alpha b + (1-\alpha)c)$$
$$\iff \quad \forall q \in \mathcal{C}: \quad \int a\, dq \geq \int b\, dq.$$

[36]The claim is easily established by induction on K.

Notation: Let $q(a) = \int a\,dq$ for any $q \in ba_1(\Sigma)$ and q-integrable function $a : \Omega \to \mathbb{R}$.

Next, some key consequences of linearity of J for the set $\mathcal{C}$ are investigated.

LEMMA 3: *Assume that J is linear. Then we can make the following statements*:

1. $p \in \mathcal{C}$ *and, for all* $q \in \mathcal{C}$, $2p - q \in \mathcal{C}$.
2. *For all* $a \in B(\Sigma)$ *such that* $a \geq 0$, *and for all* $q \in \mathcal{C}$, $2J(a) \geq q(a)$. *In particular, for all* $a, b \in B(\Sigma)$ *and all* $q \in \mathcal{C}$, $2J(|a - b|) \geq q(|a - b|) \geq |q(a) - q(b)|$.

PROOF: Consider $a, b \in B_0(\Sigma, u(X))$ such that $-a, -b, 2J(a) - a, 2J(b) - b \in B_0(\Sigma, u(X))$, so $J(a) = \frac{1}{2}I(a) - \frac{1}{2}I(-a)$ and similarly for b. Then, for all $\lambda \in (0, 1]$ and $d \in B_0(\Sigma, u(X))$, choose γ so that $\gamma - d \in B_0(\Sigma, u(X))$. Then $(1-\lambda)\gamma - \lambda a - (1-\lambda)d = \lambda(-a) + (1-\lambda)(\gamma - d) \in B_0(\Sigma, u(X))$ and similarly $(1 - \lambda)\gamma - \lambda b - (1 - \lambda)d \in B_0(\Sigma, u(X))$, so the definition of J implies that $I(\lambda a + (1-\lambda)d) = 2J(\lambda a + (1-\lambda)d) + I(\lambda(-a) + (1-\lambda)(\gamma - d)) - (1-\lambda)\gamma$ and $I(\lambda b + (1-\lambda)d) = 2J(\lambda b + (1-\lambda)d) + I(\lambda(-b) + (1-\lambda)(\gamma - d)) - (1-\lambda)\gamma$. Therefore, by linearity of J and canceling common terms, $I(\lambda a + (1-\lambda)d) \geq I(\lambda b + (1-\lambda)d)$ iff $2J(\lambda a) + I(\lambda(-a) + (1-\lambda)(\gamma - d)) \geq 2J(\lambda b) + I(\lambda(-b) + (1-\lambda)(\gamma - d))$. Since a, b were chosen so that $2J(a) - a, 2J(b) - b \in B_0(\Sigma, u(X))$, this is also equivalent to $I(\lambda(2J(a) - a) + (1-\lambda)(\gamma - d)) \geq I(\lambda(2J(b) - b) + (1-\lambda)(\gamma - d))$ by vertical invariance. Finally, since $d' \in B_0(\Sigma, u(X))$ if and only if $\gamma' - d' \in B_0(\Sigma, u(X))$ for some γ', conclude that $a \succeq b$ if and only if $2J(a) - a \succeq 2J(b) - b$. By Lemma 2, this is equivalent to the condition

$$(24)\qquad \forall q \in \mathcal{C}, \quad q(a) \geq q(b) \iff \forall q \in \mathcal{C}, \quad 2J(a) - q(a) \geq 2J(b) - q(b).$$

For arbitrary $a, b \in B_0(\Sigma)$, let $\alpha > 0$ be such that $\alpha a, \alpha b, -\alpha a, -\alpha b, 2J(\alpha a) - \alpha a, 2J(\alpha b) - \alpha b \in B_0(\Sigma, u(X))$ [such an α exists because $0 \in u(X)$]. Then Eq. (24) must hold for αa and αb, and positive homogeneity of every $q \in \mathcal{C}$ and J implies that it must hold for a and b as well.

Now, for statement 1, define $a \succeq_0 b$ for $a, b \in B_0(\Sigma, u(X))$ to mean that the left-hand side of Eq. (22) holds, as in the proof of Lemma 2. For every $q \in \mathcal{C}$, $2p(\Omega) - q(\Omega) = 1$. Furthermore, for every $E \in \Sigma$, taking $a = 1_E$ and $b = 0$, $q(E) \geq 0$ and so, by Eq. (24), $2p(E) - q(E) \geq 0$ as well. Thus, $2p - q \in ba_1(\Sigma)$. Thus, let $\mathcal{D}$ be the weak* convex closure of $\mathcal{C} \cup \{2p - q : q \in \mathcal{C}\}$. It is clear that for all $a, b \in B_0(\Sigma, u(X))$, $r(a) \geq r(b)$ for all $r \in \mathcal{D}$ implies $a \succeq_0 b$; conversely, if $a \succeq_0 b$, then $q(a) \geq q(b)$ for all $q \in \mathcal{C}$, hence $2J(a) - q(a) \geq 2J(b) - q(b)$ for all $q \in \mathcal{C}$, and hence $r(a) \geq r(b)$ for all $r \in \mathcal{D}$. Since Lemma 2 ensures that $\mathcal{C}$ is the unique set of probability charges that represents $\succeq_0$, $\mathcal{C} = \mathcal{D}$, and so for every $q \in \mathcal{C}$, $2p - q \in \mathcal{C}$ as well. This immediately implies that $p = \frac{1}{2}q + \frac{1}{2}(2p - q) \in \mathcal{C}$.

For statement 2, note first that for any $a \in B_0(\Sigma)$ with $a \geq 0$, $q(a) \geq 0$ for all $q \in \mathcal{C}$: hence, by Eq. (24), $2J(a) \geq q(a)$. The inequality now extends to $B(\Sigma)$ by sup-norm continuity of J and $q(\cdot)$. Finally, for any $a, b \in B(\Sigma)$, $2J(|a-b|) \geq q(|a-b|) \geq |q(a)-q(b)|$, where the second equality follows, for example, from Dudley (1989, Theorem 5.1.1). *Q.E.D.*

Conclude with a useful "vertical invariance" property.

LEMMA 4: *In the setting of Lemma 2, if $a, b \in B(\Sigma, u(X))$ and for some $\delta \in \mathbb{R}$, $q(a) = q(b) + \delta$ for all $q \in \mathcal{C}$, then $I(a) = I(b) + \delta$.*

PROOF: Assume first that $\inf b(\Omega), \sup b(\Omega) \in \operatorname{int}(u(X))$. Then there exists $\alpha \in (0,1)$ such that $b + \alpha\delta \in B(\Sigma, u(X))$. For all $k \geq 0$, let $a^k = [1-(1-\alpha)^k]a + (1-\alpha)^k b$. Then $a^k \in B(\Sigma, u(X))$ for all $k \geq 0$. Furthermore,

$$\begin{aligned}(1-\alpha)a^k + \alpha a &= (1-\alpha)[1-(1-\alpha)^k]a + (1-\alpha)^{k+1}b + \alpha a \\ &= [1-(1-\alpha)^{k+1}]a + (1-\alpha)^{k+1}b = a^{k+1}.\end{aligned}$$

Now write $d \simeq d'$ to signify that $I(\alpha d + (1-\alpha)c) = I(\alpha d' + (1-\alpha)c)$ for all $\alpha \in (0,1]$ and $c \in B(\Sigma, u(X))$. By Lemma 2, $d \simeq d'$ iff $q(d) = q(d')$ for all $q \in \mathcal{C}$. In particular, $\simeq$ is conic: $d \simeq d'$ implies that $\beta d + (1-\beta)d'' \simeq \beta d' + (1-\beta)d''$. Note that $\simeq$ is the symmetric part of the relation $\succeq$ defined in the proof of Lemma 2.

CLAIM 1: *For all k, $a^k + \alpha(1-\alpha)^k\delta \in B(\Sigma, u(X))$ and $a^{k+1} \simeq a^k + \alpha(1-\alpha)^k\delta$.*

PROOF: For $k = 0$, $a^0 + \alpha(1-\alpha)^0\delta = b + \alpha\delta \in B(\Sigma, u(X))$ by the choice of δ. Furthermore, for all $q \in \mathcal{C}$, $q(a^1) = q((1-\alpha)b + \alpha a) = (1-\alpha)q(b) + \alpha q(a) = (1-\alpha)q(b) + \alpha q(b) + \alpha\delta = q(b) + \alpha\delta = q(a^0 + \alpha(1-\alpha)^0\delta)$, so $a^1 \simeq a^0 + \alpha(1-\alpha)^0\delta$. By induction, for $k > 0$,

$$\begin{aligned}(1-\alpha)[a^{k-1} + \alpha(1-\alpha)^{k-1}\delta] + \alpha a &= (1-\alpha)a^{k-1} + \alpha a + \alpha(1-\alpha)^k\delta \\ &= a^k + \alpha(1-\alpha)^k\delta.\end{aligned}$$

Thus, $a^k + \alpha(1-\alpha)^k\delta \in B(\Sigma, u(X))$ because $a, a^{k-1} + \alpha(1-\alpha)^{k-1}\delta \in B(\Sigma, u(X))$. Furthermore, if $a^k \simeq a^{k-1} + \alpha(1-\alpha)^{k-1}\delta$, then also

$$\begin{aligned}a^{k+1} &= (1-\alpha)a^k + \alpha a \simeq (1-\alpha)[a^{k-1} + \alpha(1-\alpha)^{k-1}\delta] + \alpha a \\ &= a^k + \alpha(1-\alpha)^k\delta\end{aligned}$$

because $\simeq$ is conic. *Q.E.D.*

The claim implies that, for all $k \geq 1$, $I(a^k) = I(a^{k-1} + \alpha(1-\alpha)^{k-1}\delta) = I(a^{k-1}) + \alpha(1-\alpha)^{k-1}\delta$, where the second equality follows from vertical invariance. Thus,

$$I(a^k) = I(b) + \alpha\delta \sum_{\ell=0}^{k-1} (1-\alpha)^\ell = I(b) + \alpha\delta \frac{1-(1-\alpha)^k}{\alpha}$$
$$= I(b) + \delta[1-(1-\alpha)^k].$$

Since $a^k \to a$ and I is continuous, the result follows.

If b is arbitrary, for $k \geq 0$, let $a^k = \frac{k}{k+1}a$ and $b^k = \frac{k}{k+1}b$, so in particular $b^k(\Omega) \subset \operatorname{int}(u(X))$. Furthermore, for every $k \geq 0$ and $q \in C$, $q(a^k) = \frac{k}{k+1}q(a) = \frac{k}{k+1}q(b) + \frac{k}{k+1}\delta = q(b^k) + \frac{k}{k+1}\delta$, and it has just been shown that then $I(a^k) = I(b^k) + \frac{k}{k+1}\delta$. Since $a^k \to a$ and $b^k \to b$, continuity implies that $I(a) = I(b) + \delta$. *Q.E.D.*

B.3. *Monotone Continuity*

Assume that Γ is nonsingleton. A functional $H : B_0(\Sigma, \Gamma) \to \mathbb{R}$ is *monotonely continuous* iff, for every $\alpha, \beta, \gamma \in \Gamma$ with $\alpha > \beta > \gamma$ and every sequence of events $(A_k) \subset \Sigma$ such that $A_k \supset A_{k+1}$ for all n and $\bigcap A_k = \emptyset$, there is k such that $H(\alpha - (\alpha-\gamma)1_{A_k}) > \beta > H(\gamma + (\alpha-\gamma)1_{A_k})$—or, abusing the notation for binary acts, $H(\gamma A_k \alpha) > \beta > H(\alpha A_k \gamma)$.

Continue to focus on the representation I, u of $\succcurlyeq$; assume w.l.o.g. that $0 \in \operatorname{int}(u(X))$. Clearly, $\succcurlyeq$ satisfies Axiom 6 iff I is monotonely continuous. This property will now be characterized in terms of the functional J defined in Lemma 1.

LEMMA 5: *The following statements are equivalent*:
1. *I is monotonely continuous*.
2. *For every decreasing sequence $(A_k) \subset \Sigma$ such that $\bigcap A_k = \emptyset$, $J(1_{A_k}) \to 0$.*

Thus, if I is monotonely continuous, the charge p representing J is actually a measure.

PROOF OF LEMMA 5: $1 \Rightarrow 2$: Let $\alpha \in u(X)$ be such that $\alpha > 0$ and $-\alpha \in u(X)$. For every $\varepsilon \in (0, \alpha)$, there is k' such that $\varepsilon > I(\alpha 1_{A_{n'}})$ and k'' such that $I(\alpha(1 - 1_{A_{k''}})) > \alpha - \varepsilon$ (take $\gamma = 0$ and $\beta = \varepsilon, \alpha - \varepsilon$ in the definition of monotone continuity). Letting $k = \max(k', k'')$, so $A \subset A_{k'}$ and $A \subset A_{k''}$, by monotonicity both $\varepsilon > I(\alpha 1_{A_k})$ and $I(\alpha(1 - 1_{A_k})) > \alpha - \varepsilon$ hold. Furthermore, since $-\alpha \in u(X)$, vertical invariance of I implies that $I(\alpha(1 - 1_{A_k})) = \alpha + I(-\alpha 1_{A_k}) > \alpha - \varepsilon$, that is, $\varepsilon > -I(-\alpha 1_{A_k})$. Hence, $\varepsilon > \frac{1}{2}I(\alpha 1_{A_k}) - \frac{1}{2}I(-\alpha 1_{A_k}) = J(\alpha 1_{A_k})$. To sum up, if $\eta \geq 1$, then monotonicity implies that $J(1_{A_k}) \leq \eta$ for all k, and for $\eta \in (0,1)$, taking $\varepsilon = \eta\alpha$ yields k such that $J(1_{A_k}) = \frac{1}{\alpha}J(\alpha 1_{A_k}) < \frac{1}{\alpha}\varepsilon = \eta$.

$2 \Rightarrow 1$: Fix $\alpha, \beta, \gamma \in u(X)$ with $\alpha > \beta > \gamma$. Then there is k' such that $J(\gamma + (\alpha - \gamma)1_{A_{k'}}) < \gamma + \frac{1}{2}(\beta - \gamma)$. Let $\mu = \alpha + \gamma$, so $\mu - \gamma - (\alpha - \gamma)1_{A'_k} = \alpha - (\alpha - \gamma)1_{A_{k'}} \in B_0(\Sigma, u(X))$. Then, by the definition of J,

$$\gamma + \frac{1}{2}(\beta - \gamma) > \frac{1}{2}\mu + \frac{1}{2}I\bigl(\gamma + (\alpha - \gamma)1_{A_{k'}}\bigr) - \frac{1}{2}I\bigl(\mu - \gamma - (\alpha - \gamma)1_{A_{k'}}\bigr);$$

substituting for μ and simplifying this reduces to

$$\begin{aligned}\frac{1}{2}\beta &> \frac{1}{2}\alpha + \frac{1}{2}I\bigl(\gamma + (\alpha - \gamma)1_{A_{k'}}\bigr) - \frac{1}{2}I\bigl(\alpha - (\alpha - \gamma)1_{A_{k'}}\bigr)\\ &\geq \frac{1}{2}I\bigl(\gamma + (\alpha - \gamma)1_{A_{k'}}\bigr),\end{aligned}$$

where the inequality follows from monotonicity of I, as $\alpha - (\alpha - \gamma)1_{A_{k'}} \leq \alpha$. Thus, $\beta > I(\gamma + (\alpha - \gamma)1_{A_{k'}})$. Similarly, there is k'' such that $J(\alpha - (\alpha - \gamma)1_{A_{k''}}) > \alpha - \frac{1}{2}(\alpha - \beta)$, that is,

$$\alpha - \frac{1}{2}(\alpha - \beta) < \frac{1}{2}\mu + \frac{1}{2}I\bigl(\alpha - (\alpha - \gamma)1_{A_{k''}}\bigr) - \frac{1}{2}I\bigl(\mu - \alpha + (\alpha - \gamma)1_{A_{k''}}\bigr),$$

and again substituting for μ and simplifying yields

$$\begin{aligned}\frac{1}{2}\beta &< \frac{1}{2}\gamma + \frac{1}{2}I\bigl(\alpha - (\alpha - \gamma)1_{A_{k''}}\bigr) - \frac{1}{2}I\bigl(\gamma + (\alpha - \gamma)1_{A_{k''}}\bigr)\\ &\leq \frac{1}{2}I\bigl(\alpha - (\alpha - \gamma)1_{A_{k''}}\bigr),\end{aligned}$$

because $\gamma + (\alpha - \gamma)1_{A_{k''}} \geq \gamma$. Thus, $I(\alpha - (\alpha - \gamma)1_{A_{k''}}) > \beta$. Therefore, by monotonicity, $k = \max(k', k'')$ satisfies $I(\alpha - (\alpha - \gamma)1_{A_k}) > \beta > I(\gamma + (\alpha - \gamma)1_{A_k})$, as required. *Q.E.D.*

B.4. *Proof of Theorem 1*

It is clear that statement 2 implies 3 in Theorem 1; thus, focus on the nontrivial implications.

B.4.1. *Statement 3 Implies 1*

For all $a \in u \circ \mathcal{F}_0$, let $J_p(a) = \int a\,dp$ and $I(a) = J_p(a) + A(\mathrm{E}_p[\zeta a])$. Thus, for all $f, g \in \mathcal{F}_0$, $f \succcurlyeq g$ iff $I(u \circ f) \geq I(u \circ g)$. It is easy to verify that I is constant-mixture invariant and normalized (because $\mathrm{E}_p[\zeta_i] = 0$ for all i and $A(0) = 0$). Furthermore, by part 3 of Definition 1, it is monotonic and hence a niveloid by Proposition 6. This implies that $\succcurlyeq$ satisfies the first five axioms in statement 1. Furthermore, for all $a \in u \circ \mathcal{F}_0$, letting $\gamma \in u(X)$ be such that $\gamma - a \in u \circ \mathcal{F}_0$,

$$\begin{aligned}
J(a) &\equiv \frac{1}{2}\gamma + \frac{1}{2}\hat{I}(a) - \frac{1}{2}\hat{I}(\gamma - a) \\
&= \frac{1}{2}\gamma + \frac{1}{2}J_p(a) + \frac{1}{2}A(\mathrm{E}_p[\zeta a]) - \frac{1}{2}J_p(\gamma - a) \\
&\quad - \frac{1}{2}A\big(\mathrm{E}_p[\zeta(\gamma - a)]\big) \\
&= J_p(a),
\end{aligned}$$

as $\mathrm{E}_p[\zeta_i] = 0$ for all i and $A(\phi) = A(-\phi)$ for all $\phi \in \mathcal{E}(\mathcal{F}_0; p, \zeta)$; thus, the functional J defined in Lemma 1 coincides with J_p on $u \circ \mathcal{F}_0$, and hence it is affine; thus, $\succcurlyeq$ satisfies Axioms 7 and 8 as well. Moreover, since p is countably additive, if $(A^k) \subset \Sigma$ decreases to $\emptyset$, $J(1_{A^k}) = J_p(1_{A^k}) \downarrow 0$, and Lemma 5 implies that I is monotonely continuous, so $\succcurlyeq$ satisfies Axiom 6.

B.4.2. *Statement 1 Implies 2*

Since $\succcurlyeq$ satisfies Axioms 1–5, it admits a nondegenerate niveloidal representation I, u by Proposition 6. Furthermore, it is w.l.o.g. to assume that $0 \in \mathrm{int}(u(X))$. Moreover, since $\succcurlyeq$ satisfies Axioms 7 and 8, the functional J defined in Eq. (20) is affine on $u \circ \mathcal{F}_0$ by Lemma 1. Finally, since $\succcurlyeq$ satisfies Axiom 6, I is monotonely continuous, so Lemma 5 implies that the measure p representing J is countably additive. This will be the baseline prior in the VEU representation.

The next step is to construct the adjustment factors $(\zeta_i)_{0 \leq i < n}$ along the lines of Section 4.1. A slight detour and a preliminary result are needed to accommodate infinite state spaces. Let H be the Hilbert space of (p-equivalence classes of) Σ-measurable, square-integrable functions on Ω. Let $\langle a, b \rangle = \mathrm{E}_p[ab]$ for all $a, b \in H$. Recall that since Σ is countably generated, H is separable (cf., e.g., Bogachev (2007, Sections 1.12.102 and 4.7.63)).

LEMMA 6: *For every $q \in \mathcal{C}$, the map $a \mapsto \int a\,dq$ is an $L_2(p)$-continuous linear functional on H; in particular, J extends to a continuous linear functional on H. Furthermore, for each such $q \in \mathcal{C} \setminus \{p\}$, there exists $a_q \in H$ such that $\int a\,dq = \langle a, a_q \rangle$ for all $a \in H$, and $\sup_{\omega \in \Omega} |a_q(\omega)| = 2$.*

PROOF: By Lemma 3, $\int a\,dq \le 2J(a)$ for all $a \in B(\Sigma)$ such that $a \ge 0$; hence, possibly by considering truncations and taking suprema, $\int |a|^2\,dq \le 2J(|a|^2)$ for all Σ-measurable functions a, where one or both integrals may be infinite. In particular, every $a \in H$ is also square-integrable with respect to q, so $a \mapsto \int a\,dq$ is well defined on H.

Furthermore, if $a^k \to a$ in the $L_2(p)$ norm topology (i.e. $J(|a^k - a|^2) \to 0$), then clearly $q(|a^k - a|^2) \to 0$, which implies that $q(a^k) \to q(a)$.[37] Hence, $q(\cdot)$ is a continuous linear functional on H.

By the Riesz–Frechet theorem, there exists $a_q \in H$ such that $q(a) = \langle a, a_q\rangle$. I claim that a_q can be chosen to be bounded. To this end, for every $M > 0$, let $E_M = \{\omega : a_q(\omega) > M\}$. Then

$$M \cdot p(E_M) \le \int 1_{E_M} a_q\,dp = q(E_M) \le 2p(E_M),$$

where the second inequality follows from Lemma 3. Then either $p(E_M) = 0$ or $M \le 2$. Therefore, since q is positive, $0 \le a_q(\omega) \le 2$ p-a.e., so the claim follows. *Q.E.D.*

Now define the set

$$C = \{c \in H : \forall q \in \mathcal{C}, q(c) = J(c)\}. \tag{25}$$

To interpret, recall that an act f in $\mathcal{F}_0$ or $\mathcal{F}_b$ is crisp iff $\lambda f + (1-\lambda)g \sim \lambda x + (1-\lambda)g$ for all $g \in \mathcal{F}_0$, $\lambda \in (0, 1]$, and $x \in X$ such that $x \sim f$. This is equivalent to $I(\lambda u \circ f + (1-\lambda)u \circ g) = \lambda I(u \circ f) + I((1-\lambda)u \circ g)$ and, hence, by Lemma 2, to $q(u \circ f) = I(u \circ f)$ for all $q \in \mathcal{C}$. In particular, this implies $J(u \circ f) = I(u \circ f)$ by Lemma 3, and so f is crisp iff $u \circ f \in C$. The definition of the set C employs this characterization of crisp acts to identify a class of functions in H with analogous properties.

Conclude by showing that C is closed in H. By Lemma 6, if $(c^k) \subset C$ is such that $c^k \to c$ for some $c \in H$ in the $L_2(p)$ norm topology, then $J(c) = \lim_k J(c^k) = \lim_k q(c^k) = q(c)$ for all $q \in \mathcal{C}$; therefore $c \in C$.

Construction of the Adjustment Factors $(\zeta_i)_{0 \le i < n}$: Observe that $\{a_q - 1_\Omega : q \in \mathcal{C}\}$ is a subset of the separable space H and hence admits a countable dense subset $\{b_0, b_1, \dots\}$. Note that for every $i \ge 0$, $\sup_\Omega |b_i| \le 1$ by Lemma 6.

Let NC be the closure in H of the linear span of $\{b_0, b_1, \dots\}$; by Dudley (1989, Corollary 5.4.10), the Hilbert subspace NC has a countable orthonormal basis $\{\zeta_0, \zeta_1, \dots\}$, obtained by applying the Gram–Schmidt procedure to $\{b_0, b_1, \dots\}$. In particular, note that this procedure ensures that each ζ_i is bounded, that is, it is an element of $B(\Sigma)$.

[37]If $q(|a^k - a|^2) \to 0$, then a^k converges to a in the $L_2(q)$ norm. By Dudley (1989, Theorems 5.5.2 and 5.1.1), $a^k \to a$ in the $L_1(q)$ norm as well, and this implies the claim.

Consider the orthogonal complement $NC^\perp = \{c \in H : \forall b \in NC, \langle c, b\rangle = 0\}$. If $c \in C$, then $q(c) = J(c)$ for all $q \in \mathcal{C}$, so in particular $\langle c, b_i\rangle = 0$ for all $i \geq 0$. Therefore, $\langle c, b\rangle = 0$ for any b in the linear span of $\{b_0, b_1, \ldots\}$, which is the same as the linear span of $\{\zeta_0, \zeta_1, \ldots\}$. Finally, this implies that $\langle c, b\rangle = 0$ for all $b \in NC$. Thus, $C \subset NC^\perp$. Conversely, if $c \in NC^\perp$, then in particular $\langle c, a_q - 1_\Omega\rangle = 0$ for all $q \in \mathcal{C}$, that is, $q(c) = J(c)$; hence, $c \in C$. Thus, conclude that $C = NC^\perp$.

Since $1_\Omega \in C = NC^\perp$, $\langle 1_\Omega, \zeta_i\rangle = 0$, that is, $\mathrm{E}_p[\zeta_i] = 0$ for all i. Henceforth, let n denote the number of nonzero ζ_i's and assume w.l.o.g. that these are the first n elements of the sequence $\zeta_0, \zeta_1, \ldots$.

Construction of the Adjustment Function A: Define first $\tilde{I} : u \circ \mathcal{F}_b + C \to \mathbb{R}$ by letting $\tilde{I}(a + c) = I(a) + J(c)$ for all $a \in u \circ \mathcal{F}_b$ and $c \in C$. This is well posed: if $a + c = a' + c'$ for $a, a' \in u \circ \mathcal{F}_b$ and $c, c' \in C$, then $a - a' = c' - c \in C$; thus, for all $q \in \mathcal{C}$, $q(a) = q(a' + (a - a')) = q(a') + q(c' - c) = q(a') + J(c' - c)$, so that $I(a) = I(a') + J(c' - c)$ by Lemma 4. Thus, $I(a) + J(c) = I(a') + J(c' - c) + J(c) = I(a') + J(c')$, as needed. Also note that if $a \in u \circ \mathcal{F}_b$, then there exists $\gamma \in \mathbb{R}$ such that $\gamma - a \in u \circ \mathcal{F}_b$ and therefore $-a \in u \circ \mathcal{F}_b + C$ because C contains all constant functions.

Now consider $\varphi \in \mathcal{E}(u \circ \mathcal{F}_b; p, \zeta)$, so there is $a \in u \circ \mathcal{F}_b$ such that $\varphi = \mathrm{E}_p[\zeta a] = (\langle a, \zeta_i\rangle)_i$. Then $b = \sum_i \varphi_i \zeta_i$ is the projection of a onto NC, $a - b \in NC^\perp = C$, and thus $b = a + (b - a) \in u \circ \mathcal{F}_b + C$. Let $A(\varphi) = \frac{1}{2}\tilde{I}(b) + \frac{1}{2}\tilde{I}(-b)$.

To see that $A(\cdot)$ is well defined, suppose that $\varphi = \mathrm{E}_p[\zeta a']$ for some $a' \neq a$ in $u \circ \mathcal{F}_b$. Then b is also the projection of a' onto NC and $a' - b \in C$, so $b = a' + (b - a') \in u \circ \mathcal{F}_b + C$; thus, $A(\cdot)$ is well defined because so is $\tilde{I}$. Furthermore, $0_n = \mathrm{E}_p[a\zeta]$ for $a = 0$, which is the unique element in $NC \cap C$. Thus, $A(0_n) = \frac{1}{2}I(0) + \frac{1}{2}I(-0) = 0$. Finally, if $\varphi = \mathrm{E}_p[\zeta a]$ for some $a \in u \circ \mathcal{F}_b$ and $b \in NC$ is the projection of a, then $b \in u \circ \mathcal{F}_b + C$ and so $-b = \sum_i (-\varphi_i)\zeta_i \in u \circ \mathcal{F}_b + C$, which implies that $A(-\varphi) = \frac{1}{2}\tilde{I}(-b) + \frac{1}{2}\tilde{I}(b) = A(\varphi)$.

Finally, verify that the map $f \mapsto \mathrm{E}_p[u \circ f] + A(\mathrm{E}_p[\zeta u \circ f])$ indeed represents preferences. For $a \in u \circ \mathcal{F}_b$, if $\gamma - a \in u \circ \mathcal{F}_b$, then $\mathrm{E}_p[a] + A(\mathrm{E}_p[\zeta a]) = J(a) + \frac{1}{2}\tilde{I}(a) + \frac{1}{2}\tilde{I}(-a) = \frac{1}{2}\gamma + \frac{1}{2}I(a) - \frac{1}{2}I(\gamma - a) + \frac{1}{2}I(a) + \frac{1}{2}I(\gamma - a) + \frac{1}{2}J(-\gamma) = I(a)$, decomposing $-a$ as $(\gamma - a) + (-\gamma)$ with $\gamma - a \in u \circ \mathcal{F}_b$ and $-\gamma \in C$. This completes the proof.

B.4.3. *Proof of Corollary 1*

By Corollary 4, $\succcurlyeq$ has a unique extension to $\mathcal{F}_b$ that satisfies Axioms 1–5. Clearly, this preference also satisfies Axiom 6, and Lemma 1 shows that it satisfies Axioms 7 and 8 as well. The argument in the preceding subsection actually constructs a VEU representation of the extension of $\succcurlyeq$ to $\mathcal{F}_b$, which is sharp.

B.4.4. *Uniqueness*

Consider two VEU representations (u, p, n, ζ, A) and (u', p', n', ζ', A') of $\succcurlyeq$, and assume that the former is sharp. By standard arguments, $u' = \alpha u + \beta$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$. Consequently, $a \in u \circ \mathcal{F}_0$ if and only if $\alpha a + \beta \in u' \circ \mathcal{F}_0$. Next, for every $a \in u \circ \mathcal{F}_0$, let $I(a) = \mathrm{E}_p[a] + A(\mathrm{E}_p[\zeta \cdot a])$; define I' similarly using the second VEU representation. By Corollary 3, $\alpha I(a) + \beta = I'(\alpha a + \beta)$ for every $a \in u \circ \mathcal{F}_0$. If $a, \gamma - a \in u \circ \mathcal{F}_0$, then $\alpha a + \beta, \alpha(\gamma - a) + \beta \in u' \circ \mathcal{F}_0$, and so, if J and J' are the functionals defined from I and I', respectively, as in Eq. (20), then

$$\begin{aligned} J'(\alpha a + \beta) &= \alpha \frac{1}{2}\gamma + \beta + \frac{1}{2} I'(\alpha a + \beta) - \frac{1}{2} I'(\alpha(\gamma - a) + \beta) \\ &= \alpha \frac{1}{2}\gamma + \beta + \frac{1}{2}[\alpha I(a) + \beta] - \frac{1}{2}[\alpha I(\gamma - a) + \beta] \\ &= \alpha J(a) + \beta. \end{aligned}$$

This implies that linear extensions of J and J' to $B(\Sigma)$ coincide, and so $p = p'$; hence,

$$\begin{aligned} \alpha A(\mathrm{E}_p[\zeta \cdot a]) &= \alpha I(a) + \beta - \alpha J(a) - \beta = I'(\alpha a + \beta) - J'(\alpha a + \beta) \\ &= A'(\mathrm{E}_p[\zeta' \cdot \alpha a]) \end{aligned} \tag{26}$$

for all $a \in u \circ \mathcal{F}_0$, where the last equality uses the fact that $\mathrm{E}_p[\zeta'_j] = 0$ for all $0 \leq j < n'$. Now, to define a suitable linear surjection $T: \mathcal{E}(u' \circ \mathcal{F}_0; p, \zeta') \to \mathcal{E}(u \circ \mathcal{F}_0; p, \zeta)$, suppose that $\mathrm{E}_p[\zeta' \cdot \alpha a] = \mathrm{E}_p[\zeta' \cdot \alpha b]$ for $a, b \in u \circ \mathcal{F}_0$. Let $\gamma \in \mathbb{R}$ be such that $\gamma - b \in u \circ \mathcal{F}_0$, so there is $f \in \mathcal{F}_0$ such that $\frac{1}{2}a + \frac{1}{2}(\gamma - b) = u \circ f$ or, equivalently, $\frac{1}{2}(\alpha a + \beta) + \frac{1}{2}[\alpha(\gamma - b) + \beta] = \alpha u \circ f + \beta = u' \circ f$. But then $\mathrm{E}_p[\zeta' \cdot u' \circ f] = \mathrm{E}_p[\zeta' \cdot \frac{1}{2}(a - b)] = 0$, which implies that f is crisp.[38] Since (u, p, n, ζ, A) is sharp, $\mathcal{E}_p[\zeta \cdot u \circ f] = 0$ and so $\mathrm{E}_p[\zeta \cdot a] = \mathrm{E}_p[\zeta \cdot b]$. Thus, we can define T by letting $T(\mathrm{E}_p[\zeta' \cdot \alpha a]) = \mathrm{E}_p[\zeta \cdot a]$ for all $a \in u \circ \mathcal{F}_0$. That T is affine and onto is immediate. Finally, if $\varphi' = \mathrm{E}_p[\zeta' \cdot \alpha a]$, then $A(T(\varphi')) = A(T(\mathrm{E}_p[\zeta' \cdot \alpha a])) = A(\mathrm{E}_p[\zeta \cdot a]) = \frac{1}{\alpha} A'(\mathrm{E}_p[\zeta' \cdot \alpha a]) = \frac{1}{\alpha} A'(\varphi')$, where the second equality follows from the definition of T and the third equality follows from Eq. (26): thus, $A = \frac{1}{\alpha} A' \circ T$.

Finally, if (u', p, n', ζ', A') is also sharp, assume that $\mathrm{E}_p[\zeta \cdot a] = \mathrm{E}_p[\zeta \cdot b]$: arguing as above, if $\gamma - b \in u \circ \mathcal{F}_0$ and $u \circ f = \frac{1}{2}a + \frac{1}{2}(\gamma - b)$, then f is crisp. Since (u', p, n', ζ', A') is sharp, $\mathrm{E}_p[\zeta' \cdot u' \circ f] = 0$, that is, $\mathrm{E}_p[\zeta' \cdot \alpha a] = \mathrm{E}_p[\zeta' \cdot \alpha b]$. Thus, T is a bijection.

B.4.5. *Proof of Proposition 1*

Recall first that, as shown in the proof of uniqueness, $\mathrm{E}_p[\zeta \cdot a] = 0$ implies that a is a crisp function (in $u \circ \mathcal{F}_0$ or $u \circ \mathcal{F}_b$).

[38]For all $g \in \mathcal{F}_0$ and $x \in X$ with $f \sim x$, $\mathrm{E}_p[\lambda u' \circ f + (1-\lambda) u' \circ g] + A'(\mathrm{E}_p[\zeta' \cdot [\lambda u' \circ f + (1-\lambda) u' \circ g]]) = \mathrm{E}_p[\lambda u' \circ f + (1-\lambda) u' \circ g] + A'(\mathrm{E}_p[\zeta' \cdot (1-\lambda) u' \circ g]) = \mathrm{E}_p[\lambda u'(x) + (1-\lambda) u' \circ g] + A'(\mathrm{E}_p[\zeta' \cdot [\lambda u'(x) + (1-\lambda) u' \circ g]])$.

Part 1. If $n = \infty$, the statement holds vacuously. Otherwise, observe that $\mathrm{E}_p[\zeta \cdot u \circ f_1], \dots, \mathrm{E}_p[\zeta \cdot u \circ f_m]$ is a collection of $m > n$ vectors in $\mathbb{R}^n$, so there must be $\beta_1, \dots, \beta_m \in \mathbb{R}$, not all zero, such that $\sum_j \beta_j \mathrm{E}_p[\zeta \cdot u \circ f_j] = 0$. Let $\bar{\beta} = \sum_j |\beta_j| > 0$. Now define $\alpha_1, \dots, \alpha_m$ and $g_1, \dots, g_m \in \mathcal{F}_b$ by letting (a) $\alpha_j = |\beta_j|/\bar{\beta}$ and (b) $g_j = f_j$ if $\beta_j > 0$, and g_j be such that $u \circ g_j = \gamma_j - u \circ f_j$ for a suitable $\gamma_j \in \mathbb{R}$ otherwise. Then $\sum_j \alpha_j u \circ g_j = \frac{1}{\bar{\beta}} \sum_j \beta_j u \circ f_j - \frac{1}{\bar{\beta}} \sum_{j:\beta_j<0} \beta_j \gamma_j$. Therefore, by construction, $\mathrm{E}_p[\zeta \cdot \sum_j \alpha_j u \circ g_j] = 0$, so $\sum_j \alpha_j g_j$ is a crisp combination of $f_1, \dots, f_m$.

Part 2. Suppose that (u, p, n, ζ, A) is sharp, so in particular the ζ_i's are orthonormal.

Since each ζ_i is bounded, there exists $\gamma > 0$ such that $\gamma \zeta'_i \in u \circ \mathcal{F}_b$ for all $i = 0, \dots, m-1$; thus, for each such i, let $f_i \in \mathcal{F}_b$ be such that $u \circ f_i = \gamma \zeta'_i$.

Suppose there exists a crisp combination $\sum_j \alpha_j g_j$ of $f_0, \dots, f_{m-1}$. For j such that $g_j \neq f_j$, suppose that $u \circ g_j = \gamma_j - u \circ f_j$. Also, for all $j = 0, \dots, m-1$, let $\beta_j = \alpha_j$ if $g_j = f_j$ and let $\beta_j = -\alpha_j$ otherwise. Then, since (u, p, n, ζ', A') is sharp, $\mathrm{E}_p[\zeta' \cdot \sum_j \beta_j \zeta'_j] = \frac{1}{\gamma} \mathrm{E}_p[\zeta \cdot \sum_j \alpha_j u \circ g_j] = 0_n$, where constants cancel because $\mathrm{E}_p[\zeta'] = 0_n$. But since $\zeta'_0, \dots, \zeta'_{m-1}$ are orthonormal, $\mathrm{E}_p[\zeta'_i \cdot \sum_{j=0}^{m-1} \beta_j \zeta'_j] = \beta_i$ for $0 \leq i < m-1$, and not all β_i's are zero: contradiction.

Part 3. Suppose that (u', p', n', ζ', A') is another representation of $\succcurlyeq$ on $\mathcal{F}_b$ and, by contradiction, $n' < n$. By part 2, there is a tuple $f_0, \dots, f_{n'}$ that admits no crisp combination; however, by part 1, every tuple of $n' + 1$ elements *must* contain a crisp combination: contradiction. Thus, $n' \geq n$.

Part 4. If: This part follows from part 2 and the fact that if $\succcurlyeq$ is not EU, then $n > 0$.

Only if: Since (u, p, n, ζ, A) is sharp and $n = 1$, $\succcurlyeq$ is not EU. Now suppose that f, g, $\bar{g}$, and α are such that both $\alpha f + (1-\alpha) g$ and $\alpha f + (1-\alpha)\bar{g}$ are crisp. Since the representation is sharp, $\mathrm{E}_p[\zeta \cdot u \circ [\alpha f + (1-\alpha) g]] = \mathrm{E}_p[\zeta \cdot u \circ [\alpha f + (1-\alpha)\bar{g}]] = 0$; hence, for all $\bar{f}$ such that $(f, \bar{f})$ are complementary, also $\mathrm{E}_p[\zeta \cdot u \circ [\alpha \bar{f} + (1-\alpha) g]] = \mathrm{E}_p[\zeta \cdot u \circ [\alpha \bar{f} + (1-\alpha)\bar{g}]] = 0$. This implies that there is a tuple of size $m = 2$ that admits no crisp combinations, which contradicts part 2. *Q.E.D.*

B.5. *Ambiguity Aversion*

PROOF OF COROLLARY 2: If $\succcurlyeq$ satisfies Ambiguity Aversion, then I is concave (cf. MMR, p. 28); in particular, if $a, \gamma - a \in u \circ \mathcal{F}_0$,

$$\begin{aligned}\frac{1}{2}\gamma &= I\left(\frac{1}{2}a + \frac{1}{2}(\gamma - a)\right) \geq \frac{1}{2}I(a) + \frac{1}{2}I(\gamma - a)\\ &= \frac{1}{2}\int a\,dp + \frac{1}{2}A(\mathrm{E}_p[\zeta \cdot a]) + \frac{1}{2}\gamma\end{aligned}$$

$$-\frac{1}{2}\int a\,dp+\frac{1}{2}A\big(\mathrm{E}_p[\zeta\cdot(\gamma-a)]\big)$$

$$=\frac{1}{2}\gamma+A(\mathrm{E}_p[\zeta\cdot a]),$$

and so A is nonpositive. Finally, A is clearly also concave.

Conversely, suppose that A is concave (hence, also nonpositive). Then I is concave, so for all $f, g\in\mathcal{F}_0$ with $f\sim g$, $I(u\circ[\lambda f+(1-\lambda)g])\geq I(u\circ\lambda f)$. *Q.E.D.*

PROOF OF PROPOSITION 2: That condition 3 $\Rightarrow$ 1 is immediate (consider the EU preference determined by p and u). To see that condition 3 $\Leftrightarrow$ 2, note that if $f,\bar{f}$ are complementary, with $\frac{1}{2}f+\frac{1}{2}\bar{f}\sim z\in X$, $f\sim x$, and $\bar{f}\sim\bar{x}$, then $\frac{1}{2}f+\frac{1}{2}\bar{f}\succcurlyeq\frac{1}{2}x+\frac{1}{2}\bar{x}$ iff

$$\begin{aligned}u(z)&\geq\frac{1}{2}\int u\circ f\,dp+\frac{1}{2}A(\mathrm{E}_p[\zeta\cdot u\circ f])\\&\quad+\frac{1}{2}\int u\circ\bar{f}\,dp+\frac{1}{2}A(\mathrm{E}_p[\zeta\cdot u\circ\bar{f}])\\&=u(z)+A(\mathrm{E}_p[\zeta\cdot u\circ f]),\end{aligned}$$

because $\mathrm{E}_p[\zeta\cdot u\circ\bar{f}]=-\mathrm{E}_p[\zeta\cdot u\circ f]$ and A is symmetric; hence, the required ranking obtains iff $A(\mathrm{E}_p[\zeta\cdot u\circ f])\leq 0$.

Turn now to condition 1 $\Rightarrow$ 3. Suppose that $\succcurlyeq$ is more ambiguity-averse than some EU preference relation $\succcurlyeq'$. By Corollary B.3 in Ghirardato, Maccheroni, and Marinacci (2004), one can assume that $\succcurlyeq'$ is represented by the nonconstant utility u on X. Arguing by contradiction, suppose that there is $f\in\mathcal{F}_0$ such that $A(\mathrm{E}_p[\zeta\cdot u\circ f])>0$. Let $\gamma\in\mathbb{R}$ be such that $\gamma-u\circ f\in B_0(\Sigma,u(X))$ and let $\bar{f}\in\mathcal{F}_0$ be such that $u\circ\bar{f}=\gamma-u\circ f$. Then $A(\mathrm{E}_p[\zeta\cdot u\circ\bar{f}])=A(\mathrm{E}_p[\zeta\cdot u\circ f])>0$. Furthermore, $\frac{1}{2}u\circ f+\frac{1}{2}u\circ\bar{f}=u\circ(\frac{1}{2}f+\frac{1}{2}\bar{f})=\frac{1}{2}\gamma$, which implies $A(\mathrm{E}_p[\zeta\cdot u\circ(\frac{1}{2}f+\frac{1}{2}\bar{f})])=A(0)=0$. If now $f\sim x$ and $\bar{f}\sim\bar{x}$ for $x,\bar{x}\in X$, then $\frac{1}{2}u(x)+\frac{1}{2}u(\bar{x})=\frac{1}{2}\gamma+A(\mathrm{E}_p[\zeta\cdot u\circ f])>\frac{1}{2}\gamma$, so $\frac{1}{2}x+\frac{1}{2}\bar{x}\succ\frac{1}{2}f+\frac{1}{2}\bar{f}$. Now let $z\in X$ be such that $\frac{1}{2}f(\omega)+\frac{1}{2}\bar{f}(\omega)\sim z$ for all ω; then $\frac{1}{2}x+\frac{1}{2}\bar{x}\succ z$, so $\frac{1}{2}x+\frac{1}{2}\bar{x}\succ' z$. But $f\sim x$ and $\bar{f}\sim\bar{x}$ imply $f\succcurlyeq' x$ and $\bar{f}\succcurlyeq'\bar{x}$, and since $\succcurlyeq'$ is an EU preference, $\frac{1}{2}f+\frac{1}{2}\bar{f}\succcurlyeq'\frac{1}{2}x+\frac{1}{2}\bar{x}$; hence, $z\succcurlyeq'\frac{1}{2}x+\frac{1}{2}\bar{x}$, a contradiction.

To see that condition 3 $\Leftrightarrow$ 4, consider first the following claim.

CLAIM 2: *For a complementary pair* $(f,\bar{f})$ *such that* $f\sim\bar{f}$, $\frac{1}{2}f+\frac{1}{2}\bar{f}\sim z\succcurlyeq f$ *iff* $A(\mathrm{E}_p[\zeta\cdot u\circ f])\leq 0$.

PROOF: To prove this claim, let $\frac{1}{2}f+\frac{1}{2}\bar{f}\sim z\in X$. Then, since $f\sim\bar{f}$ and these acts have the same adjustments, $\int u\circ f\,dp=\int u\circ\bar{f}\,dp$, so both integrals equal $u(z)$. Therefore, $\frac{1}{2}f + \frac{1}{2}\bar{f} \sim z \succcurlyeq f$ if and only if $u(z) \geq u(z) + A(\mathrm{E}_p[\zeta \cdot u \circ f]) = \int u \circ f\, dp + A(\mathrm{E}_p[\zeta \cdot u \circ f])$. *Q.E.D.*

Claim 2 immediately shows that condition 3 implies 4. For the converse, assume that Axiom 11 holds and consider the cases (a) $\succcurlyeq$ satisfies Certainty Independence or (b) $u(X)$ is unbounded. In case (a), then I is positively homogeneous, so if $\varphi = \mathrm{E}_p[\zeta \cdot a]$ for some $a \in B(\Sigma, u(X))$ and $\alpha > 0$, then $A(\alpha\varphi) = \hat{I}(\alpha a) - J(\alpha a) = \alpha[\hat{I}(a) - J(a)] = \alpha A(\varphi)$; that is, A is also positively homogeneous. In this case, it is w.l.o.g. to assume that $u(X) \supset [-1, 1]$ and prove the result for $f \in \mathcal{F}_0$ such that $\|u \circ f\| \leq \frac{1}{3}$. This ensures the existence of $\bar{f} \in \mathcal{F}_0$ such that $u \circ \bar{f} = -u \circ f$, as well as $g, \bar{g} \in \mathcal{F}_0$ such that $u \circ g = u \circ f - \int u \circ f\, dp$ and $u \circ \bar{g} = u \circ \bar{f} - \int u \circ \bar{f}\, dp = -u \circ g$. By construction, $(g, \bar{g})$ are complementary and $g \sim \bar{g}$, because $\int u \circ g\, dp = \int u \circ \bar{g}\, dp = 0$. Claim 2 implies that $A(\mathrm{E}_p[\zeta \cdot u \circ f]) = A(\mathrm{E}_p[\zeta \cdot u \circ g]) \leq 0$, as required.

In case (b), suppose $u(X)$ is unbounded below (the other case is treated analogously). Consider $f \in \mathcal{F}_0$ and construct $\bar{f} \in \mathcal{F}_0$ such that $u \circ \bar{f} = \min u \circ f(\Omega) + \max u \circ f(\Omega) - f$. Then f and $\bar{f}$ are complementary. If $f \sim \bar{f}$, then Claim 2 suffices to prove the result. Otherwise, let $\delta = \int u \circ f\, dp - \int u \circ \bar{f}\, dp$. If $\delta > 0$, consider $f' \in \mathcal{F}_0$ such that $u \circ f' = u \circ f - \delta$: then $\int u \circ f'\, dp = \int u \circ \bar{f}\, dp$, and f' and $\bar{f}$ are complementary, so $f' \sim \bar{f}$ and Claim 2 implies that $A(\mathrm{E}_p[\zeta \cdot u \circ f]) = A(\mathrm{E}_p[\zeta \cdot u \circ f']) \leq 0$. If instead $\delta < 0$, consider f' such that $u \circ f' = \bar{f} - \delta$, so again $f \sim f'$ and Claim 2 can be invoked to yield the required conclusion. *Q.E.D.*

PROOF OF PROPOSITION 3: Statement 2 $\Rightarrow$ 1 is immediate, so focus on statement 1 $\Rightarrow$ 2. Since constant acts are complementary, assume w.l.o.g. that $u^1 = u^2 \equiv u$; it is also w.l.o.g. to assume that $0 \in \mathrm{int}(X)$. Next, consider $a \in u \circ \mathcal{F}_0$ such that $-a \in u \circ \mathcal{F}_0$ and let $f, \bar{f}$ be such that $a = u \circ f$ and $-a = u \circ \bar{f}$. Then, by the properties of the VEU representation, $f \succcurlyeq_1 \bar{f}$ iff $f \succcurlyeq_2 \bar{f}$ is equivalent to $\mathrm{E}_{p^1}[a] \geq 0$ iff $\mathrm{E}_{p^2}[a] \geq 0$. By positive homogeneity, this is true for all $a \in B_0(\Sigma)$; in particular, $\mathrm{E}_{p^1}[a - \mathrm{E}_{p^1}[a]] = 0$, so $\mathrm{E}_{p^2}[a - \mathrm{E}_{p^1}[a]] = 0$, that is, $\mathrm{E}_{p^1}[a] = \mathrm{E}_{p^2}[a]$ for all $a \in B_0(\Sigma)$ and the claim follows.

Now suppose that statements 1 and 2 hold, and that the VEU representations under consideration are sharp. Then an act f is crisp for $\succcurlyeq_j$ iff $\mathrm{E}_p[\zeta^j \cdot u \circ f] = 0$. Thus, if $\zeta^1 = \zeta^2$, $\succcurlyeq_1$ and $\succcurlyeq_2$ admit the same crisp acts. Conversely, suppose $\succcurlyeq_1$ and $\succcurlyeq_2$ admit the same crisp acts; then, for all $a \in u \circ \mathcal{F}_0$, $\mathrm{E}_p[\zeta^1 \cdot a] = 0$ iff $\mathrm{E}_p[\zeta^2 \cdot a] = 0$, and by positive homogeneity the same is true for all $a \in B_0(\Sigma)$. Therefore, if $\mathrm{E}_p[\zeta^1 \cdot a] = \mathrm{E}_p[\zeta^1 \cdot b]$ for $a, b \in u \circ \mathcal{F}_0$, then also $\mathrm{E}_p[\zeta^2 \cdot a] = \mathrm{E}_p[\zeta^2 \cdot b]$ and the converse implication also holds. Hence, we can define $\bar{A}^2 : \mathcal{E}(u \circ \mathcal{F}_0; p, \zeta^1) \to \mathbb{R}$ by $\bar{A}^2(\mathrm{E}_p[\zeta^1 \cdot a]) = A^2(\mathrm{E}_p[\zeta^2 \cdot a])$ to get a new VEU representation $(u, p, n^1, \zeta^1, \bar{A}^2)$ for $\succcurlyeq_2$. *Q.E.D.*

PROOF OF PROPOSITION 4: Suppose that $\succsim_1$ is more ambiguity-averse than $\succsim_2$. Pick $f \in \mathcal{F}_0$ and let $x \in X$ be such that

$$u(x) = \mathrm{E}_p[u \circ f] + A^1(\mathrm{E}_p[\zeta^1 \cdot u \circ f]).$$

Then $f \succsim_2 x$ and therefore

$$\begin{aligned} &\mathrm{E}_p[u \circ f] + A^2(\mathrm{E}_p[\zeta^2 \cdot u \circ f]) \geq u(x) \\ &\quad = \mathrm{E}_p[u \circ f] + A^1(\mathrm{E}_p[\zeta^1 \cdot u \circ f]), \end{aligned}$$

which yields the required inequality.

Conversely, suppose

$$A^1(\mathrm{E}_p[\zeta^1 \cdot u \circ f]) \leq A^2(\mathrm{E}_p[\zeta^2 \cdot u \circ f])$$

for all $f \in \mathcal{F}_0$. Then, for all $x \in X$, $f \succsim_1 x$ implies

$$\begin{aligned} \mathrm{E}_p[u \circ f] + A^2(\mathrm{E}_p[\zeta^2 \cdot u \circ f]) &\geq \mathrm{E}_p[u \circ f] + A^1(\mathrm{E}_p[\zeta^1 \cdot u \circ f]) \\ &\geq u(x), \end{aligned}$$

that is, $f \succsim_2 x$, as required. The final claim is immediate. *Q.E.D.*

B.6. *Updating*

For $a, b \in u \circ \mathcal{F}_0$, let $aEb \in u \circ \mathcal{F}_0$ be the function that equals a on E and equals b elsewhere.

PROOF OF REMARK 1: Only if: It will be shown that, for any event $E \in \Sigma$, $p(E) = 0$ implies $I(a) = I(b)$ for all $a, b \in u \circ \mathcal{F}_0$ such that $a(\omega) = b(\omega)$ for $\omega \notin E$. To see this, assume w.l.o.g. that $I(a) \geq I(b)$ and let $\alpha = \max\{\max a(\Omega), \max b(\Omega)\}$ and $\beta = \min\{\min a(\Omega), \min b(\Omega)\}$. Then monotonicity implies that $I(\alpha Ea) \geq I(a) \geq I(b) \geq I(\beta Eb) = I(\beta Ea)$. Thus, it is sufficient to show that $I(\alpha Ea) = I(\beta Ea)$. This is immediate if $\alpha = \beta$, so assume $\alpha > \beta$. Since $p(E) = 0$, $\mathrm{E}_p[\alpha Ea] = \mathrm{E}_p[1_{\Omega \setminus E} a] = \mathrm{E}_p[\beta Ea]$, so if $I(\alpha Ea) > I(\beta Ea)$, it must be the case that $A(\mathrm{E}_p[\zeta \cdot \alpha Ea]) > A(\mathrm{E}_p[\zeta \cdot \beta Ea])$. Letting $\gamma = \alpha + \beta$, as usual $\gamma - \alpha Ea, \gamma - \beta Ea \in u \circ \mathcal{F}_0$. Now

$$\begin{aligned} I(\gamma - \alpha Ea) &= \mathrm{E}_p[[\gamma - \alpha Ea]] + A\big(\mathrm{E}_p[\zeta \cdot [\gamma - \alpha Ea]]\big) \\ &= \mathrm{E}_p\big[1_{\Omega \setminus E}[\gamma - a]\big] + A(-\mathrm{E}_p[\zeta \cdot \alpha Ea]) \\ &= \mathrm{E}_p\big[1_{\Omega \setminus E}[\gamma - a]\big] + A(\mathrm{E}_p[\zeta \cdot \alpha Ea]) \\ &> \mathrm{E}_p\big[1_{\Omega \setminus E}[\gamma - a]\big] + A(\mathrm{E}_p[\zeta \cdot \beta Ea]) \\ &= \mathrm{E}_p\big[1_{\Omega \setminus E}[\gamma - a]\big] + A\big(\mathrm{E}_p[\zeta \cdot [\gamma - \beta Ea)]\big) \end{aligned}$$

$$= \mathrm{E}_p[\gamma - \beta Ea] + A\big(\mathrm{E}_p[\zeta \cdot [\gamma - \beta Ea)]\big)$$
$$= I(\gamma - \beta Ea),$$

which is a violation of monotonicity, as $\gamma - \alpha = \beta < \alpha = \gamma - \beta$.

If: Suppose that $p(E) > 0$ and fix $x, y \in X$ with $x \succ y$. If $xEy \succ y$, we are done. Otherwise, note that $xEy \sim y$, that is, $[u(x) - u(y)]p(E) + A([u(x) - u(y)]\mathrm{E}_p[\zeta \cdot 1_E]) = 0$, implies

$$\begin{aligned} A\big([u(x) - u(y)]\mathrm{E}_p\big[\zeta \cdot 1_{\Omega\setminus E}\big]\big) &= A\big([u(x) - u(y)]\mathrm{E}_p[\zeta \cdot 1_E]\big) \\ &= -[u(x) - u(y)]p(E); \end{aligned}$$

hence,

$$\begin{aligned} I(u \circ yEx) &= u(y) + p(\Omega \setminus E)[u(x) - u(y)] \\ &\quad + A\big([u(x) - u(y)]\mathrm{E}_p\big[\zeta \cdot 1_{\Omega\setminus E}\big]\big) \\ &= u(y) + p(\Omega \setminus E)[u(x) - u(y)] - [u(x) - u(y)]p(E) \\ &= u(y) + [u(x) - u(y)][p(\Omega \setminus E) - p(E)] < u(x), \end{aligned}$$

because $p(\Omega \setminus E) - p(E) = 1 - 2p(E) < 1$ as $p(E) > 0$. Thus, $x \succ yEx$ and again Axiom 12 holds. *Q.E.D.*

PROOF OF PROPOSITION 5: Since E is not null, $p(E) > 0$, so $p(\cdot|E)$ is well defined.

CLAIM 3: *If $(f, \bar{f})$ are complementary and constant on $\Omega \setminus E$, then*

$$\frac{1}{2}f + \frac{1}{2}\bar{f}(\omega) \sim \frac{1}{2}\bar{f} + \frac{1}{2}f(\omega)$$

holds if and only if $u(f(\omega)) = \mathrm{E}_p[u \circ f] = \mathrm{E}_p[u \circ f|E]$ for all $\omega \in \Omega \setminus E$.

PROOF: Let $\gamma \in \mathbb{R}$ be such that $\frac{1}{2}\gamma = \frac{1}{2}u(f(\omega)) + \frac{1}{2}u(\bar{f}(\omega))$ for all $\omega \in \Omega$. Also let $\alpha = u(f(\omega))$ and $\beta = u(\bar{f}(\omega))$ for any (hence all) $\omega \in \Omega \setminus E$. Then $u \circ \bar{f} = \gamma - u \circ f$ and $\beta = \gamma - \alpha$. Thus, for $\omega \in \Omega \setminus E$,

$$\begin{aligned} &I\left(u \circ \left(\frac{1}{2}f + \frac{1}{2}\bar{f}(\omega)\right)\right) \\ &\quad = \frac{1}{2}\mathrm{E}_p[u \circ f] + \frac{1}{2}\beta + A\left(\frac{1}{2}\mathrm{E}_p[\zeta \cdot u \circ f]\right) \\ &\quad = \frac{1}{2}\mathrm{E}_p[u \circ f] + \frac{1}{2}\gamma - \frac{1}{2}\alpha + A\left(\frac{1}{2}\mathrm{E}_p[\zeta \cdot u \circ f]\right) \end{aligned}$$

and

$$
\begin{aligned}
&I\left(u\circ\left(\frac{1}{2}\bar{f}+\frac{1}{2}f(\omega)\right)\right)\\
&\quad=\frac{1}{2}\mathrm{E}_p[u\circ\bar{f}]\,dp+\frac{1}{2}\alpha+A\left(\frac{1}{2}\mathrm{E}_p[\zeta\cdot u\circ\bar{f}]\right)\\
&\quad=\frac{1}{2}\gamma-\mathrm{E}_p[u\circ f]+\frac{1}{2}\alpha+A\left(\frac{1}{2}\mathrm{E}_p[\zeta\cdot u\circ f]\right),
\end{aligned}
$$

where the last equality uses the fact that $\mathrm{E}_p[\zeta\cdot u\circ\bar{f}]=-\mathrm{E}_p[\zeta\cdot u\circ f]$ and A is symmetric. Hence, $\frac{1}{2}f+\frac{1}{2}\bar{f}(\omega)\sim\frac{1}{2}\bar{f}+\frac{1}{2}f(\omega)$ holds if and only if $\alpha=\mathrm{E}_p[u\circ f]$. Furthermore, $\mathrm{E}_p[u\circ f]=\mathrm{E}_p[u\circ f\cdot 1_E]+\alpha p(\Omega\setminus E)$, so it follows that $\alpha=\mathrm{E}_p[u\circ f|E]$ as well. *Q.E.D.*

Next, note that the adjustment factors $\zeta_E=(\zeta_{i,E})_{0\le i<n}$ defined by Eq. (7) are easily seen to be bounded and to have zero mean. Also observe that

$$
\begin{aligned}
\mathrm{E}_p[\zeta_E\cdot a|E]&=p(E)\big\{\mathrm{E}_p[\zeta\cdot a|E]-\mathrm{E}_p[a|E]\mathrm{E}_p[\zeta_i|E]\big\}\\
&=\mathrm{E}_p[\zeta\cdot a\cdot 1_E]-\mathrm{E}_p[a|E]\mathrm{E}_p[\zeta_i\cdot 1_E]\\
&=\mathrm{E}_p\big[\zeta\cdot aE(\mathrm{E}_p[a|E])\big],
\end{aligned}
\tag{27}
$$

where the last equality follows from $-\mathrm{E}_p[\zeta_i\cdot 1_E]=\mathrm{E}_p[\zeta_i\cdot 1_{\Omega\setminus E}]$. To show that (u,p,n,ζ_E,A) is a VEU representation, it is sufficient to verify monotonicity. Observe that for $a,b\in u\circ\mathcal{F}_0$, $a\ge b$ implies that $\mathrm{E}_p[a|E]\ge\mathrm{E}_p[b|E]$, and hence $aE(\mathrm{E}_p[a|E])\ge bE(\mathrm{E}_p[b|E])$. Since (u,p,n,ζ,A) is a VEU representation, $\mathrm{E}_p[aE(\mathrm{E}_p[a|E])]+A(\mathrm{E}_p[\zeta\cdot aE(\mathrm{E}_p[a|E])])\ge\mathrm{E}_p[bE(\mathrm{E}_p[b|E])]+A(\mathrm{E}_p[\zeta\cdot bE(E_p[b|E]))$, that is, by Eq. (27), $\mathrm{E}_p[a|E]+A(\mathrm{E}_p[\zeta_E\cdot a|E])\ge\mathrm{E}_p[b|E]+A(\mathrm{E}_p[\zeta_E\cdot b|E])$, as required.

Now suppose part 1 holds. Fix $f,g,\bar{f},\bar{g}\in\mathcal{F}_0$ as in Axiom 14. By Claim 3, $u\circ f(\omega)=\mathrm{E}_p[u\circ f|E]=\mathrm{E}_p[u\circ f]$ and $u\circ g(\omega)=\mathrm{E}_p[u\circ g|E]=\mathrm{E}_p[u\circ g]$ for all $\omega\in\Omega\setminus E$. Then the axiom implies that $f\succcurlyeq_E g$ iff $f\succcurlyeq g$, that is, iff

$$
\begin{aligned}
&\mathrm{E}_p[u\circ f]+A(\mathrm{E}_p[\zeta\cdot u\circ f])\ge\mathrm{E}_p[u\circ g]+A(\mathrm{E}_p[\zeta\cdot u\circ g])\\
&\quad\Leftrightarrow\quad \mathrm{E}_p[u\circ f|E]+A\big(\mathrm{E}_p\big[\zeta\cdot u\circ fE(\mathrm{E}_p[u\circ f|E])\big]\big)\\
&\qquad\qquad\ge\mathrm{E}_p[u\circ g|E]+A\big(\mathrm{E}_p\big[\zeta\cdot u\circ gE(\mathrm{E}_p[u\circ g|E])\big]\big)\\
&\quad\Leftrightarrow\quad \mathrm{E}_p[u\circ f|E]+A(\mathrm{E}_p[\zeta_E\cdot u\circ f|E])\\
&\qquad\qquad\ge\mathrm{E}_p[u\circ g|E]+A(\mathrm{E}_p[\zeta_E\cdot u\circ g|E]).
\end{aligned}
$$

If now $f,g\in\mathcal{F}_0$ are arbitrary, let $x,y\in X$ be such that $u(x)=\mathrm{E}_p[u\circ f|E]$ and $u(y)=\mathrm{E}_p[u\circ g|E]$. Notice that then $\mathrm{E}_p[u\circ fEx]=\mathrm{E}_p[u\circ fEx|E]=u(x)$, and similarly for gEy. Finally, let f' and g' be such that (fEx, f') and (gEy, g') are complementary; notice that this requires that f' and g' be constant on $\Omega \setminus E$. Then, by Claim 3, the acts fEx, f', gEy, and g' satisfy the assumptions of Axiom 14, and the argument just given shows that then $fEx \succcurlyeq_E gEy$ iff $\mathrm{E}_p[u \circ f|E] + A(\mathrm{E}_p[\zeta_E \cdot u \circ f|E]) \geq \mathrm{E}_p[u \circ g|E] + A(\mathrm{E}_p[\zeta_E \cdot u \circ g|E])$. But by Axiom 13, $fEx \succcurlyeq_E gEy$ iff $f \succcurlyeq_E g$, so part 2 holds.

In the opposite direction, assume that part 2 holds. It is then immediate that Axiom 13 is satisfied. Now assume that f, g, $\bar{f}$, and $\bar{g}$ are as in Axiom 14. Then Claim 3 shows that $u(f(\omega)) = \mathrm{E}_p[u \circ f|E]$ and $u(g(\omega)) = \mathrm{E}_p[u \circ g|E]$ for all $\omega \in \Omega \setminus E$, so

$$\begin{aligned}
&\mathrm{E}_p[u \circ f|E] + A(\mathrm{E}_p[\zeta_E \cdot u \circ f|E]) \\
&\quad = p(E)\mathrm{E}_p[u \circ f|E] + p(\Omega \setminus E)u(f(\omega)) \\
&\qquad + A\big(\mathrm{E}_p\big[p(E)(\zeta - \mathrm{E}_p[\zeta|E])u \circ f|E\big]\big) \\
&\quad = \mathrm{E}_p[u \circ f] + A\big(\mathrm{E}_p[\zeta 1_E u \circ f] + \mathrm{E}_p\big[\zeta 1_{\Omega\setminus E}\big]\mathrm{E}_p[u \circ f|E]\big) \\
&\quad = \mathrm{E}_p[u \circ f] + A(\mathrm{E}_p[\zeta u \circ f]),
\end{aligned}$$

and similarly for g, so Axiom 14 holds. *Q.E.D.*

Conclude by verifying that the "law of iterated conditioning" holds: with notation as in Section 4.4,

$$\begin{aligned}
\zeta_{i,E,F} &= p(F|E) \cdot [\zeta_{i,E} - \mathrm{E}_p[\zeta_{i,E}|F]] \\
&= p(F|E) \\
&\quad \cdot \big[p(E) \cdot (\zeta_i - \mathrm{E}_p[\zeta_i|E]) - \mathrm{E}_p\big[p(E)(\zeta_i - \mathrm{E}_p[\zeta_i|E])|F\big]\big] \\
&= p(F)\zeta_i - p(F)\mathrm{E}_p[\zeta_i|E] - p(F)\mathrm{E}_p[\zeta_i|F] + p(F)\mathrm{E}_p[\zeta_i|E] \\
&= \zeta_{i,F}.
\end{aligned}$$

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