Ambiguity in the small and in the large: Online Appendix

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June 2012

A Locally Lipschitz preferences

We consider a preference $\succcurlyeq$ that admits a monotonic, continuous, normalized, Bernoullian representation (I, u), and introduce a novel axiom that is equivalent to the assertion that I is locally Lipschitz.[1] Recall that $x_h \in X$ denotes the certainty equivalent of act $h \in \mathscr{F}$.

Axiom 1 (Locally Bounded Improvements) *For every $h \in \mathscr{F}^{\text{int}}$, there are $y \in X$ and $g \in \mathscr{F}$ with $g(s) \succ h(s)$ for all s such that, for all $(h^n) \subset \mathscr{F}$ and $(\lambda^n) \subset [0,1]$ with $h^n \to h$ and $\lambda^n \downarrow 0$,*

$$\lambda^n g + (1-\lambda^n)h^n \prec \lambda^n y + (1-\lambda^n)x_{h^n} \quad \textit{eventually.}$$

To gain intuition, focus on the constant sequence with $h^n = h$. Since preferences are Bernoullian, the individual's evaluation of $\lambda y + (1-\lambda)x_h$ changes linearly with λ. On the other hand, her evaluation of $\lambda g + (1-\lambda)h$ may improve in arbitrary non-linear (though continuous) ways as λ increases from 0 to 1 (recall that g is pointwise preferred to h). The Axiom states that, when λ is close to 0, this improvement is comparable to the *linear* change in preference that applies to

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[1]That is: for every $a \in \text{int } B_0(\Sigma, u(X))$, there are $\epsilon > 0$ and $L > 0$ such that $|I(b) - I(c)| \le L\|b - c\|$ for all $b, c \in B_0(\Sigma, u(X))$ with $\|b - a\| < \epsilon$ and $\|c - a\| < \epsilon$.

$\lambda y + (1-\lambda)x_h$ (which may still be very rapid, if y is 'much' preferred to x_h). Hence, it imposes a bound on the instantaneous rate of change in preferences, as a function of λ. Furthermore, this bound is required to be uniform in a neighborhood of h.

Proposition 1 *Let $\succcurlyeq$ be a preference that admits a monotonic, continuous, Bernoullian, normalized representation (I,u). Then $\succcurlyeq$ satisfies Axiom 1 if and only if I is locally Lipschitz in the interior of its domain.*

Proof: (If): Functionally, the displayed equation in Axiom 1 is equivalent to

$$I(\lambda^n[u\circ g - u\circ h^n] + u\circ h^n) = I(\lambda^n u\circ g + (1-\lambda^n)u\circ h^n) < I(\lambda^n u(y) + (1-\lambda^n)u(x^n)) =$$
$$= \lambda^n u(y) + (1-\lambda^n)u(x^n) = \lambda^n[u(y) - I(u\circ h^n)] + I(u\circ h^n). \tag{1}$$

Notice that the second equality uses the assumption that I is normalized. Since $u\circ h^n \to u\circ h$ in the sup norm, for every $\epsilon \in \big(0, \min_s[u(g(s)) - u(h(s))]\big)$, and for n large enough, $\max_s|u(h(s)) - u(h^n(s))| < \min_s[u(g(s)) - u(h(s))] - \epsilon$, so that, for every s, $u(h^n(s)) = u(h(s)) + [u(h^n(s)) - u(h(s))] < u(h(s)) + \min_{s'}[u(g(s')) - u(h(s'))] - \epsilon \le u(h(s)) + u(g(s)) - u(h(s)) - \epsilon = u(g(s)) - \epsilon$. In other words, $u(g(s)) - u(h^n(s)) > \epsilon$ for all s and all n large enough. Moreover, for n large enough, $\lambda^n\epsilon + h^n \in B_0(\Sigma, u(X))$. Since I is monotonic, and rearranging terms,

$$\frac{I(\lambda^n\epsilon + u\circ h^n) - I(u\circ h^n)}{\lambda^n} < u(y) - I(u\circ h^n) \quad \text{eventually.}$$

Again because $u\circ h^n \to u\circ h$, eventually $I(u\circ h^n) \ge I(u\circ h) - \epsilon$, so finally

$$\frac{I(\lambda^n\epsilon + u\circ h^n) - I(u\circ h^n)}{\lambda^n} < u(y) - I(u\circ h) + \epsilon \quad \text{eventually.}$$

This implies that, for a suitable $\epsilon > 0$, $I^\circ(u\circ h;\epsilon) \le u(y) - I(u\circ h) + \epsilon < \infty$.

To sum up, for every h such that $u\circ h \in \operatorname{int} B_0(\Sigma, u(X))$, there are $\epsilon > 0$ and $y \in X$ such that $I^\circ(u\circ h;\epsilon) \le u(y) - I(u\circ h) + \epsilon < \infty$. Since I is monotonic, by Proposition 4 in Rockafellar (1980), I is directionally Lipschitzian; by Theorem 3 therein, the Clarke-Rockafeller derivative of I in the direction a at $u\circ h$, denoted $I^\uparrow(u\circ h; a)$, equals $\liminf_{b\to a} I^\circ(u\circ h; b)$. Since $I^\circ(u\circ h;\cdot)$ is monotonic because I is, this implies that, for all a such that $a(s) < \epsilon$, $I^\uparrow(u\circ h; a) \le I^\circ(u\circ h;\epsilon) < \infty$. Therefore, the constant function 0 is in the interior of $\{a : I^\uparrow(u\circ h; a) < \infty\}$. Again by Theorem 3 in Rockafellar (1980), this implies that I is directionally Lipschitz with respect to the vector 0; as noted on p. 267 therein, it is 'an easy fact to verify' that this is equivalent to the assertion that I is locally Lipschitz at $u \circ h$.

(Only if): Conversely, suppose I is Lipschitz near $u \circ h$. Since h is interior, I is monotonic and normalized, and $I^{\circ}(u \circ h; \cdot)$ is continuous, there is $\epsilon > 0$ such that $I^{\circ}(u \circ h; \epsilon) < u(y) - I(u \circ h) - \epsilon$ for some $y \in X$. Then, for all $(h^n) \to h$ and $(\lambda^n) \downarrow 0$, eventually

$$\frac{I(\lambda^n[\epsilon + u \circ h^n] + (1-\lambda^n) u \circ h^n) - I(u \circ h^n)}{\lambda^n} = \frac{I(\lambda^n \epsilon + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h) - \epsilon.$$

Now choose n large enough so that $\max_s |u(h(s)) - u(h^n(s))| < \frac{\epsilon}{2}$. Then a fortiori, for every s, $u(h(s)) - u(h^n(s)) < \frac{\epsilon}{2}$, i.e. $u(h(s)) < u(h^n(s)) + \frac{\epsilon}{2}$, and therefore $u(h(s)) + \frac{\epsilon}{2} < u(h^n(s)) + \epsilon$. Because h is interior, there is $\delta \in (0, \frac{\epsilon}{2}]$ such that $u \circ h + \delta = u \circ g$ for some $g \in \mathscr{F}$; for such g, the above argument implies that $u(g(s)) < u(h^n(s)) + \epsilon$ for all s, and of course $g(s) \succ h(s)$ for all s. By monotonicity, conclude that, for all n sufficiently large,

$$\frac{I(\lambda^n u \circ g + (1-\lambda^n) u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h) - \epsilon.$$

Finally, by choosing n large enough, we can ensure that $I(u \circ h^n) < I(u \circ h) + \epsilon$, and therefore

$$\frac{I(\lambda^n u \circ g + (1-\lambda^n) u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h^n).$$

Rearranging terms yields Eq. (1), so the axiom holds. ■

B Nice MBL preferences

Proposition 2 *A monotonic, isotone and concave function $I : B_0(\Sigma, \Gamma) \to \mathbb{R}$ (for some interval Γ) is nice everywhere in the interior of its domain.*

Proof: Recall that a monotone concave I is locally Lipschitz; furthermore, ∂I coincides with the superdifferential of I (e.g. Rockafellar, 1980, p. 278), and it is monotone, in the sense that

$$\forall c, c' \in \text{int } B_0(\Sigma, \Gamma),\ Q \in \partial I(c), Q' \in \partial I(c'), \qquad Q(c - c') \leq Q'(c - c').^{2} \tag{2}$$

[2]Since ∂I is the superdifferential of I, $Q(c' - c) \geq I(c') - I(c)$ and $Q'(c - c') \geq I(c) - I(c')$. Summing these inequalities yields the inequality in the text.

Fix $c' \in \text{int } B_0(\Sigma,\Gamma)$ and suppose that $Q_0 \in \partial I(c')$. Then, for every $c \in \text{int } B_0(\Sigma,\Gamma)$ and every $Q \in \partial I(c)$, $Q(c-c') \leq 0$. Since c' is interior, the set $\hat{\Gamma} = \Gamma \cap \{\gamma \in \mathbb{R} : \gamma > c'(s) \; \forall s\}$ is non-empty. Morevoer, for any $\gamma \in \hat{\Gamma}$, and for all $Q \in \partial I(1_S\gamma)$, $Q(1_S\gamma - c') \leq 0$. But since $\gamma - c'(s) > 0$ for all s, and I is monotonic, this requires that $\partial I(1_S\gamma) = \{Q_0\}$ for all $\gamma \in \hat{\Gamma}$.

In particular, pick $\alpha, \beta \in \hat{\Gamma}$, with $\alpha > \beta$. Since I is isotone, $I(1_S\alpha) > I(1_S\beta)$. By the mean-value theorem (Lebourg, 1979), there must be $\mu \in (0,1)$ and $Q \in \partial I(\mu 1_S\alpha + (1-\mu)1_S\beta) = \partial I([\mu\alpha + (1-\mu)\beta]1_S)$ such that $I(1_S\alpha) - I(1_S\beta) = Q(1_S\alpha - 1_S\beta) = Q(1_S)(\alpha - \beta)$. But $\mu\alpha + (1-\mu)\beta \in \hat{\Gamma}$, so $Q = Q_0$, and therefore $I(1_S\alpha) = I(1_S\beta)$: contradiction. Therefore, I must be nice at c. ■

We now provide an axiom for MBL preferences that ensures niceness. There are obvious similarities with Axiom 1.

Axiom 2 (Non-Negligible Worsenings at h) *There are $y \in X$ with $y \prec h$ and $g \in \mathscr{F}$ with $g(s) \prec h(s)$ for all s such that, for all $(h^n) \subset \mathscr{F}$ and $(\lambda^n) \subset [0,1]$ with $h^n \to h$ and $\lambda^n \downarrow 0$,*

$$\lambda^n g + (1-\lambda^n)h^n \prec \lambda^n y + (1-\lambda^n)x_{h^n} \quad \textit{eventually.}$$

This axiom rules out the possibility that preferences may be 'flat' when moving from h toward pointwise less desirable acts g. We argue as for Axiom 1: the individual's evaluation of $\lambda y + (1-\lambda)x_h$ changes linearly with λ, whereas her evaluation of $\lambda g + (1-\lambda)h$ may worsen in arbitrary non-linear ways as λ increases from 0 to 1. Axiom 2 states that, when λ is close to 0, this worsening is comparable to the *linear* decrease in preference that applies to $\lambda y + (1-\lambda)x_h$ (which may still be very slow, if y is 'almost' as good as x_h).

Mas-Colell (1977) characterizes preferences over consumption bundles (i.e. on $\mathbb{R}^n_+$) represented by a (locally) Lipschitz and 'regular' utility function; his notion of regularity is related to niceness (cf. p. 1411); for instance, if utility is continuously differentiable, the requirement is that its gradient be non-vanishing on $\mathbb{R}^n_{++}$. Mas-Colell's axiom is not directly related to ours.

Proposition 3 *Let $\succcurlyeq$ be an MBL preference with representation (I,u), and assume that I is normalized. Then $\succcurlyeq$ satisfies Axiom 2 at $h \in \mathscr{F}^{\text{int}}$ if and only if I is nice at $u \circ h$.*

Proof: (If): As in the proof of Proposition 1, for $g, y, (h^n), (\lambda^n)$ as in the axiom,

$$I(\lambda^n[u \circ g - u \circ h^n] + u \circ h^n) < \lambda^n[u(y) - I(u \circ h^n)] + I(u \circ h^n) \quad \text{eventually.}$$

For n large, $\|u\circ h^n - u\circ h\| < 1$ and therefore $u(h^n(s)) - u(g(s)) = [u(h^n(s)) - u(h(s))] + u(h(s)) - u(g(s)) < 1 + \max_s [u(h(s)) - u(g(s))] \equiv \delta$. Since $h(s) \succ g(s)$ for all s, $\delta > 0$. Furthermore, as $n \to \infty$, eventually $\lambda^n(-\delta) + u \circ h^n \in B_0(\Sigma, u(X))$, and so, by monotonicity of I,

$$I(\lambda^n(-\delta) + u \circ h^n) < \lambda^n [u(y) - I(u \circ h^n)] + I(u \circ h^n) \quad \text{eventually.}$$

Rearranging,

$$\frac{I(\lambda^n(-\delta) + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h^n) \quad \text{eventually.}$$

Since $h^n \to h$ and I is continuous, for every $\epsilon > 0$, eventually $I(u \circ h^n) \geq I(u \circ h) - \epsilon$, and so

$$\frac{I(\lambda^n(-\delta) + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h) + \epsilon \quad \text{eventually.}$$

Therefore, $I^0(u \circ h; -\delta) \leq u(y) - I(u \circ h) + \epsilon$. Since this is true for all $\epsilon > 0$, $I^0(u \circ h; -\delta) \leq u(y) - I(u \circ h) < 0$, as $y \prec h$. But since $I^0(u \circ h; -\delta) = \max_{Q \in \partial I(u \circ h)} (-\delta) Q(S) = -\delta \min_{Q \in \partial I(u \circ h)} Q(S)$, and every $Q \in \partial I(u \circ h)$ is a positive measure because I is monotonic, the zero measure Q_0 cannot belong to $\partial I(u \circ h)$.

(Only if): Conversely, suppose I is nice at $u \circ h$. Since h is interior, there is $\delta > 0$ such that $u \circ h - \delta = u \circ g$ for some $g \in \mathscr{F}^{\text{int}}$. Since $Q_0 \notin \partial I(u \circ h)$ and I is monotonic, $I^0(u \circ h; -\frac{1}{2}\delta) < 0$. Hence, for all sequences $\lambda^n \to 0$ and $h^n \to h$ (acts), and for all $\epsilon \in (0, -I^0(u \circ h; -\frac{1}{2}\delta))$, eventually

$$\frac{I(\lambda^n(-\frac{1}{2}\delta) + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < -\epsilon.$$

In particular, find $y \in X$ such that $y \prec h$ and $I(u \circ h) - u(y) < -\frac{1}{2} I^0(u \circ h; -\frac{1}{2}\delta)$, which is possible because h is interior. Add $-\frac{1}{2} I^0(u \circ h; -\frac{1}{2}\delta)$ on both sides of this inequality to conclude that $I(u \circ h) - u(y) - \frac{1}{2} I^0(u \circ h; -\frac{1}{2}\delta) < -I^0(u \circ h; -\frac{1}{2}\delta)$, and so eventually

$$\frac{I(\lambda^n(-\frac{1}{2}\delta) + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h) + \frac{1}{2} I^0(u \circ h; -\frac{1}{2}\delta).$$

Also, for n large, $I(u(h^n)) \leq I(u(h)) - \frac{1}{2} I^0(u \circ h; -\frac{1}{2}\delta)$; conclude that, eventually,

$$\frac{I(\lambda^n(-\frac{1}{2}\delta) + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h^n).$$

Rewriting yields

$$I(\lambda^n [-\tfrac{1}{2}\delta + u \circ h^n] + (1 - \lambda^n) u \circ h^n) < \lambda^n [u(y) - I(u \circ h^n)] + I(u \circ h^n) \quad \text{eventually.}$$

Finally, if n is large enough, $\|u\circ h^n - u\circ h\| < \frac{1}{2}\delta$, so for all s, $-\frac{1}{2}\delta + u(h^n(s)) = -\frac{1}{2}\delta + u(h(s)) + [u(h^n(s)) - u(h(s))] > -\delta + u(h(s)) = u(g(s))$. Hence, finally, monotonicity implies

$$I(\lambda^n u\circ g + (1-\lambda^n) u\circ h^n) < \lambda^n u(y) - (1-\lambda^n) I(u\circ h^n) \quad \text{eventually,}$$

as required. ■

C Calculations for Example 4

Since I is continuously differentiable, it is 'strictly differentiable': see Clarke (1983, Corollary to Prop. 2.2.1). In particular, for all $e\in B_0(\Sigma)$, $h^n\to h$ and $\lambda^n\downarrow 0$, $(\lambda^n)^{-1}\left[I(\lambda^n e + (1-\lambda^n)h^n) - I((1-\lambda^n)h^n)\right] \to \nabla I(h)\cdot e$. Hence, if $\nabla I(h)\cdot f > \nabla I(h)\cdot g$, then for all sequences $\lambda^n\downarrow 0$, $h^n\downarrow 0$, eventually $(\lambda^n)^{-1}\left[I(\lambda^n f + (1-\lambda^n)h^n) - I((1-\lambda^n)h^n)\right] > (\lambda^n)^{-1}\left[I(\lambda^n g + (1-\lambda^n)h^n) - I((1-\lambda^n)h^n)\right]$, so Eq. (7) will hold for n large: hence, in this case $f \succcurlyeq^*_h g$. This is in particular the case if $h_1 > h_2 \geq 0$.

To analyze cases 2 and 3 in the text, note first that, for any pair $f,g\in\mathscr{F}$, using the formula for the difference of two cubes, $f\succcurlyeq g$ iff

$$\sum_{i=1,2}[P^i\cdot(f-g)]\;\left[(P^i\cdot f)^2 + (P^i\cdot g)^2 + (P^i\cdot f)(P^i\cdot g)\right] \geq 0. \tag{3}$$

Now consider $\epsilon, f, g, f_\epsilon, g_\epsilon$ as in the text. The rankings $\lambda^n f_\epsilon + (1-\lambda^n)h^n \succcurlyeq \lambda^n g_\epsilon + (1-\lambda^n)h^n$ and $\lambda^n f_\epsilon + (1-\lambda^n)k^n \succcurlyeq \lambda^n g_\epsilon + (1-\lambda^n)k^n$ are then equivalent to

$$\begin{aligned}
\textstyle\sum_{i=1,2} P^i\cdot\lambda^n[1+2\epsilon,-1+2\epsilon]\Big\{\;& \left[P^i\cdot\lambda^n[3+\epsilon,1+\epsilon]+\gamma\right]^2 + \left[P^i\cdot\lambda^n[2-\epsilon,2-\epsilon]+\gamma\right]^2 + \quad (4)\\
&+\left[P^i\cdot\lambda^n[3+\epsilon,1+\epsilon]+\gamma\right]\left[P^i\cdot\lambda^n[2-\epsilon,2-\epsilon]+\gamma\right]\;\Big\}\geq 0,\\
\textstyle\sum_{i=1,2} P^i\cdot\lambda^n[1+2\epsilon,-1+2\epsilon]\Big\{\;& \left[P^i\cdot\lambda^n[2+\epsilon,2+\epsilon]+\gamma\right]^2 + \left[P^i\cdot\lambda^n[1-\epsilon,3-\epsilon]+\gamma\right]^2 + \quad (5)\\
&+\left[P^i\cdot\lambda^n[2+\epsilon,2+\epsilon]+\gamma\right]\left[P^i\cdot\lambda^n[1-\epsilon,3-\epsilon]+\gamma\right]\;\Big\}\geq 0.
\end{aligned}$$

In case 3 ($\gamma=0$), divide Eqs. (4) and (5) by $(\lambda^n)^3$ and set $\epsilon=0$ to obtain the conditions

$$\begin{aligned}
&(2p-1)\left[(1+2p)^2+4+2(1+2p)\right]+(1-2p)\left[(1+2(1-p))^2+4+2(1+2(1-p))\right]\geq 0,\\
&(2p-1)\left[4+(1+2(1-p))^2+2(1+2(1-p))\right]+(1-2p)\left[4+(1+2p)^2+2(1+2p)\right]\geq 0
\end{aligned}$$

and by inspection the l.h.s. of the second inequality is the negative of the l.h.s. of the first. Furthermore, the l.h.s of the first condition equals $(2p-1)[(1+2p)^2-(1+2(1-p))^2+4(2p-1)] > 0$, because $p > \frac{1}{2}$. Therefore, for any n, when $\epsilon = 0$, Eq. (4) holds as a strict inequality, whereas the inequality in Eq. (5) fails. Hence, the same is true for any n when ϵ is positive but small. Thus. $f_\epsilon \not\succcurlyeq^*_h g_\epsilon$ for any $\epsilon \geq 0$ if $h = [0,0]$.

In case 2 ($\gamma > 0$), first take $\epsilon = 0$. We claim that Eqs. (4) and (5) can both hold only if they are in fact equalities. To see this, note that $P^1 \cdot [\alpha, \beta] = P^2 \cdot [\beta, \alpha]$ for any $\alpha, \beta \in \mathbb{R}$; hence, when $\epsilon = 0$ and $h = [\gamma, \gamma]$, the l.h.s. of Eq. (5) can be rewritten as

$$\sum_{i=1,2} P^{3-i}\cdot\lambda^n[-1,1] \left\{ \left[P^{3-i}\cdot\lambda^n[2,2]+\gamma\right]^2 + \left[P^{3-i}\cdot\lambda^n[3,1]+\gamma\right]^2 + \left[P^{3-i}\cdot\lambda^n[2,2]+\gamma\right]\left[P^{3-i}\cdot\lambda^n[3,1]+\gamma\right] \right\}.$$

It is apparent that this is the negative of the l.h.s of Eq. (4) when $\epsilon = 0$ and $h = [\gamma, \gamma]$, except that we first use P^2 and then P^1, rather than the opposite as in Eq. (4). This proves the claim.

Next, we claim that Eq. (4) holds as a strict inequality, which proves the assertion in the text that $f \not\succcurlyeq^*_h g$. Since $p > \frac{1}{2}$ and $\gamma > 0$, the first and third terms in braces are strictly greater for $i = 1$ than for $i = 2$. Since $P^2 \cdot [1,-1] = -P^1 \cdot [1,1]$, the l.h.s. of Eq. (4) is the difference of these terms, multiplied by $P^1 \cdot \lambda^n[1,-1] > 0$, and hence it is strictly positive.

Finally, if $\epsilon > 0$, and since $h = [\gamma, \gamma]$, we have $\nabla I(h)\cdot(f+\epsilon) = \nabla I(h)\cdot f + \nabla I(h)\cdot\epsilon = \nabla I(h)\cdot g + \nabla I(h)\cdot\epsilon > \nabla I(h)\cdot g - \nabla I(h)\cdot\epsilon = \nabla I(h)\cdot(g-\epsilon)$, which, as noted above, implies that $f_\epsilon \succcurlyeq^*_h g_\epsilon$.

As noted in Footnote 10, here $\partial I(0)$ contains *only* the zero vector. However, consider the monotonic, locally Lipschitz functional $J : \mathbb{R}^2 \to \mathbb{R}$ given by $J(h) = \min(I(h), h_1 + I(h))$. Then $J(h) = I(h)$ for $h \in \mathbb{R}^2$ with $h_1 \geq 0$, and $\partial J(0) = \{[\gamma, 0] : \gamma \in [0,1]\}$ (Clarke, 1983, Theorem 2.5.1). Since all mixtures in Eq. (8) are non-negative when $h \in \mathbb{R}^2_+$ and $\epsilon < 1$, even if g is replaced with $g - \epsilon$ (cf. the definition of k^n), the analysis in Example 4 applies verbatim to J. In particular, for all $\epsilon \in [0,1)$, now $f + \epsilon \succ_{C(0)} g - \epsilon$, but $f + \epsilon \not\succcurlyeq^*_0 g - \epsilon$ (the argument in the second paragraph of Ex. 4 does not apply because J is not (continuously) differentiable at 0).

D Relevant priors: a behavioral test

We conclude by showing that, given an interior act h, whether a probability $P \in ba_1(\Sigma)$ belongs to the set $C(h)$ can be ascertained without invoking Theorems 6 or 7; indeed, using only the

DM's preferences. For the result we need a notion of lower certainty equivalent of an act f for the incomplete, discontinuous preference $\succcurlyeq_h^*$ (cf. the definition of $C^*(f)$ in GMM, p. 158).

Definition 1 For any act $f \in \mathscr{F}$, a **local lower certainty equivalent** of f at $h \in \mathscr{F}^{\text{int}}$ is a prize $\underline{x}_{f,h} \in X$ such that, for all $y \in X$, $y \prec \underline{x}_{f,h}$ implies $f \succcurlyeq_h^* y$ and $y \succ \underline{x}_{f,h}$ implies $f \not\succcurlyeq_h^* y$.

Furthermore, fix $P \in ba_1(\Sigma)$ and $f \in \mathscr{F}$, and suppose that $f = \sum_{i=1}^n x_i 1_{E_i}$ for a collection of distinct prizes $x_1, \dots, x_n$ and a measurable partition $E_1, \dots, E_n$ of S. Then, define

$$x_{P,f} \equiv P(E_1)x_1 + \ldots + P(E_n)x_n.$$

That is, $x_{P,f} \in X$ is a mixture of the prizes $x_1, \dots, x_n$ delivered by f, with weights given by the probabilities that P assigns to each event $E_1, \dots, E_n$. We then have:

Corollary 4 *For any $P \in ba_1(\Sigma)$ and $h \in \mathscr{F}^{\text{int}}$ such that I is nice at $u \circ h$, $P \in C(h)$ if and only if, for all $f \in \mathscr{F}^{\text{int}}$, $\underline{x}_{f,h} \preccurlyeq x_{P,f}$.*

Proof: We show that $u(\underline{x}_{f,h}) = \min_{P \in C(h)} P(u \circ f)$; thus, the condition in the Corollary states that P satisfies $P(u \circ f) \geq \min_{P' \in C(h)} P'(u \circ f)$ for all interior f, so by linearity $P(a) \geq \min_{P' \in C(h)} P(a)$ for all $a \in B_0(\Sigma)$, and $P \in C(h)$ then follows from standard arguments.

If $\underline{x}_{f,h}$ is as in Def. 1, then $\min_{P \in C(h)} P(u \circ f) \geq u(y)$ for all $y \prec \underline{x}_{f,h}$ by (1) in Theorem 6, and so $\min_{P \in C(h)} P(u \circ f) \geq u(\underline{x}_{f,h})$. Conversely, for every y with $u(y) < \min_{P \in C(h)} P(u \circ f)$, there are $\epsilon > 0$, $y' \in X$, and $f' \in \mathscr{F}$ with $u(y') = u(y) + \epsilon$, $u \circ f' = u \circ f - \epsilon$ and $u(y') \leq \min_{P \in C(h)} P(u \circ f')$; then, by (2) in Theorem 7, since (f, y) is a spread of (f', y'), $f \succcurlyeq_h^* y$. This implies that $y \preccurlyeq \underline{x}_{f,h}$. Hence, $\min_{P \in C(h)} P(u \circ f) \leq u(\underline{x}_{f,h})$ as well. ■

E Additional properties of $\succcurlyeq_h^*$

In addition to agreeing with $\succcurlyeq$ on X, provided $\partial I(u \circ h) \neq \{Q_0\}$, $\succcurlyeq_h^*$ satisfies the following additional properties.

Lemma 5 *The preference $\succcurlyeq_h^*$ is a monotonic, independent preorder.*

Proof: Monotonicity and reflexivity are immediate from monotonicity of $\succcurlyeq$. Transitivity is immediate from the definition of $\succcurlyeq^*_h$ and transitivity of $\succcurlyeq$. It remains to be shown that $\succcurlyeq^*_h$ is independent: that is, for all $k \in \mathscr{F}$ and $\mu \in (0,1]$, $f \succcurlyeq^*_h g$ iff $\mu f + (1-\mu)k \succcurlyeq^*_h \mu g + (1-\mu)k$. Note that

$$\lambda^n[\mu f + (1-\mu)k] + (1-\lambda^n)h^n = (\lambda^n\mu)f + [1-(\lambda^n\mu)]\left\{\frac{\lambda^n(1-\mu)}{1-(\lambda^n\mu)}k + \frac{1-\lambda^n}{1-(\lambda^n\mu)}h^n\right\} \equiv$$
$$\equiv \bar{\lambda}^n f + (1-\bar{\lambda}^n)\bar{h}^n$$

with $(\bar{\lambda}^n) \downarrow 0$ and $(\bar{h}^n) \to h$, and similarly for g. Hence, if $f \succcurlyeq^*_h g$, then eventually $\bar{\lambda}^n f + (1-\bar{\lambda}^n)\bar{h}^n \succcurlyeq \bar{\lambda}^n g + (1-\bar{\lambda}^n)\bar{h}^n$; repeating the argument for all $(\lambda^n), (h^n)$ implies that $\mu f + (1-\mu)k \succcurlyeq^*_h \mu g + (1-\mu)k$. Conversely, if $\mu f + (1-\mu)k \succcurlyeq^*_h \mu g + (1-\mu)k$, define $\tilde{\lambda}^n, \tilde{h}^n$ so that

$$\tilde{\lambda}^n[\mu f + (1-\mu)k] + (1-\tilde{\lambda}^n)\tilde{h}^n = \lambda^n f + (1-\lambda^n)h^n:$$

this requires $\tilde{\lambda}^n = \frac{\lambda^n}{\mu}$, which is in $[0,1]$ for n large and converges to zero as $n \to \infty$, and

$$u \circ \tilde{h}^n = \frac{(1-\lambda^n)u \circ h^n - \tilde{\lambda}^n(1-\mu)u \circ k}{1-\tilde{\lambda}^n},$$

which is in $B_0(\Sigma, u(X))$ for n large (recall that h is interior), and indeed such that $\tilde{h}^n \to h$. Note that $\tilde{\lambda}^n, \tilde{h}^n$ do not depend on f. Again, for n large $\tilde{\lambda}^n[\mu f + (1-\mu)k] + (1-\tilde{\lambda}^n)\tilde{h}^n \succcurlyeq \tilde{\lambda}^n[\mu g + (1-\mu)k] + (1-\tilde{\lambda}^n)\tilde{h}^n$, and therefore by construction $\lambda^n f + (1-\lambda^n)h^n \succcurlyeq \lambda^n g + (1-\lambda^n)h^n$, and so, repeating for all sequences, $f \succcurlyeq^*_h g$. ■

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