Risk Sharing in the Small and in the Large
ONLINE APPENDIX

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E ONLINE APPENDIX

E.1 Calculations for the example in Section 2.2

We first briefly discuss continuity and monotonicity with respect to first-order stochastic dominance. If the set $\mathscr{D}$ of CDFs is endowed with the topology of weak convergence of measures, then continuity follows immediately from the assumption that g is continuous. Next, note that, for every CDF H, since u is a positive affine transformation of $-\frac{1}{1+x}$, it takes values in a bounded interval; therefore, $\int u^- dH > -\infty$ and $\int u^+ dH < \infty$, where as usual $u^- = \min(u, 0)$ and $u^+ = \max(u, 0)$. Now consider $F, G \in \mathscr{D}$ such that $F(x) \leq G(x)$ for all $x \geq 0$, i.e., F first-order stochastically dominates G. Then, Theorem 2.1 in Brumelle and Vickson (1975) implies that $\int u dF \geq \int u dG$ and $\int x dF \geq \int x dG$. Since g is strictly increasing, $W(F) \geq W(G)$, as required.

Now turn to risk aversion. Assume that G is a mean-preserving spread of F, in the sense that $\int_0^x [F(t) - G(t)] dt \leq 0$ for all $x \geq 0$, and $\int x dF = \int x dG$. Theorem 2.3 in Brumelle and Vickson (1975) then implies that $\int u dF \geq \int u dG$. In this case, again because g is strictly

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increasing, $W(F) \geq W(G)$.

Finally, note that, if g and u are both differentiable, for any interior bundle f,

$$\frac{\partial V}{\partial f_s}(f) = g'(\sum_t u(f_t)P_t)u'(f_s)P_s + g'(P \cdot f)P_s.$$

If $f = 1_S x$ for some $x > 0$, then

$$\frac{\partial V}{\partial f_s}(f) = g'(u(x))u'(x)P_s + g'(x)P_s = \left[g'(u(x))u'(x) + g'(x)\right]P_s,$$

i.e., $\partial V(1_S x) = \{\nabla V(1_S x)\} = \left[g'(u(x))u'(x) + g'(x)\right] \cdot P$, as asserted.

E.2 Calculations for the examples in Sections 2.1 and 2.3

We first verify that the specification of adjustment factors and function in Section 2.3, together with a uniform baseline prior, ensures strong monotonicity. The same argument applies to the simpler specification in Section 2.1; we only indicate the minor, required modifications.

We use Eq. (19): first, note that

$$\frac{\partial A}{\partial \phi_j} = -\frac{1}{2}\theta \cdot \frac{2\theta^{-1}\phi_j}{1+\theta^{-1}\phi_j^2} = -\frac{\phi_j}{1+\theta^{-1}\phi_j^2}. \tag{1}$$

Hence,

$$\left|\frac{\partial A}{\partial \phi_j}\right| = \frac{|\phi_j|}{1+\theta^{-1}\phi_j^2} = \frac{|\phi_j|}{1+\theta^{-1}|\phi_j|^2}.$$

Letting $t = |\phi_j|$, this is less then one iff $t < 1 + \theta^{-1}t^2$, i.e. iff $t^2 - \theta t + \theta > 0$. We study the function $t \mapsto t^2 - \theta t + \theta$ for $t \geq 0$. If $t = 0$, the function takes the value θ, so we need $\theta > 0$. The derivative of this function at any $t > 0$ (which is also the right derivative at 0) is $2t - \theta$, which shows that this function is strictly convex and has a minimum at $t = \frac{1}{2}\theta$, where it is equal to $\frac{1}{4}\theta^2 - \frac{1}{2}\theta^2 + \theta$. This is strictly positive iff $-\frac{1}{4}\theta + 1 > 0$, i.e. iff $\theta < 4$, as claimed.

Now consider states $s = s_1, s_2$. Only ζ_0 has non-zero values, and $\zeta_0(s) \in \{1, -1\}$. Therefore, if $\theta \in (0, 4)$,

$$1 - \frac{\phi_0}{1+\theta^{-1}\phi_0^2}\zeta_0(s) - \frac{\phi_1}{1+\theta^{-1}\phi_1^2}\zeta_1(s) \geq 1 - \left|\frac{\phi_0}{1+\theta^{-1}\phi_0^2}\right| > 0.$$

Similarly, in states $s = s_3, s_4$, $\zeta_0(s) = 0$ and $\zeta_1(s) \in \{1, -1\}$, so

$$1 - \frac{\phi_0}{1+\theta^{-1}\phi_0^2}\zeta_0(s) - \frac{\phi_1}{1+\theta^{-1}\phi_1^2}\zeta_1(s) \geq 1 - \left|\frac{\phi_1}{1+\theta^{-1}\phi_1^2}\right| > 0.$$

Act	$P(\zeta_0 u \circ f^k)$	$P(\zeta_1 u \circ f^k)$	Adjustment (omitting $\frac{1}{2}\theta$)
f^1	$\alpha - 1$	α	$-\log(1+\theta^{-1}(\alpha-1)^2) - \log(1+\theta^{-1}\alpha^2)$
f^2	0	1	$-\log(1+\theta^{-1})$
f^3	-1	0	$-\log(1+\theta^{-1})$
f^4	$-\alpha$	$1-\alpha$	$-\log(1+\theta^{-1}\alpha^2) - \log(1+\theta^{-1}(1-\alpha)^2)$

Table 2: Adjustments

so I is strictly increasing.

To adapt the argument to the preferences in Section 2.1, consider only states s_1 and s_2.

We now show that, if θ increases, the resulting preference is more GM-ambiguity-averse. By the characterization result in Siniscalchi (2009), it suffices to show that $A(\phi)$ is decreasing in θ for every ϕ. Differentiating $A(\phi)$ with respect to θ,

$$\frac{\partial A(\phi)}{\partial \theta} = -\frac{1}{2}\sum_j \log(1+\theta^{-1}\phi_j^2) - \frac{1}{2}\theta \sum_j \frac{1}{1+\theta^{-1}\phi_j^2}(-\theta^{-2}\phi_j^2);$$

it suffices to show that, for every j and ϕ_j, $\log(1+\theta^{-1}\phi_j^2) > \frac{\theta^{-1}\phi_j^2}{1+\theta^{-1}\phi_j^2}$. Let $t \equiv \theta^{-1}\phi_j^2$, so we need to show that $\log(1+t) > \frac{t}{1+t}$. Both functions equal zero at $t=0$. For $t>0$, the derivatives of the lhs and rhs are $\frac{1}{1+t}$ and $\frac{1\cdot(1+t)-t(1)}{(1+t)^2} = \frac{1}{(1+t)^2}$ respectively. Since $(1+t)^2 > 1+t$ for $t>0$, $\frac{1}{1+t} < \frac{1}{(1+t)^2}$, and therefore, for all $t>0$, $\log(1+t) = \int_0^t \frac{1}{1+s}ds > \int_0^t \frac{1}{(1+s)^2}ds = \frac{t}{1+t}$, as claimed.

We finally turn to the analysis of the specific parameterization in Section 2.3. The four acts $f^1, \ldots, f^4$ have the same expected baseline utility: $P(u \circ f^k) = 2\alpha + 1$ for $k = 1, \ldots, 4$. Hence, their ranking is entirely determined by the adjustment terms $A(P(\zeta_0 u \circ f^k), P(\zeta_i u \circ f^k))$.

These are displayed in Table 2.

In order to generate the preferences $f^1 \prec f^2$, we need to ensure that $A(P(\zeta_0 u \circ f^1), P(\zeta_1 u \circ f^1)) < A(P(\zeta_0 u \circ f^2), P(\zeta_1 u \circ f^2))$. Notice that, since $(\alpha-1)^2 = (1-\alpha)^2$, this will also ensure that $A(P(\zeta_0 u \circ f^3), P(\zeta_1 u \circ f^3)) > A(P(\zeta_0 u \circ f^4), P(\zeta_1 u \circ f^4))$ and therefore $f^3 \succ f^4$, as the adjustments for f^1 and f^2 are the same as the adjustments for f^4 and f^3 respectively. Thus, we require

$$-\log(1+\theta^{-1}(\alpha-1)^2) - \log(1+\theta^{-1}\alpha^2) < -\log(1+\theta^{-1}).$$

We now derive a condition on θ that ensures that the above inequality holds.

$$
\begin{aligned}
& -\log(1+\theta^{-1}(\alpha-1)^2)-\log(1+\theta^{-1}\alpha^2) < -\log(1+\theta^{-1}) \\
\Leftrightarrow \quad & (1+\theta^{-1}(1-\alpha)^2)(1+\theta^{-1}\alpha^2) > 1+\theta^{-1} \Leftrightarrow \quad 1+\theta^{-1}(1-\alpha)^2+\theta^{-1}\alpha^2+\theta^{-2}(1-\alpha)^2\alpha^2 > 1+\theta^{-1} \\
\Leftrightarrow \quad & (1-\alpha)^2+\alpha^2+\theta^{-1}(1-\alpha)^2\alpha^2 > 1 \\
\Leftrightarrow \quad & \theta^{-1} > \frac{1-\alpha^2-(1-\alpha^2)}{\alpha^2(1-\alpha)^2} = \frac{1-\alpha^2-1-\alpha^2+2\alpha}{\alpha^2(1-\alpha)^2} = \frac{2\alpha(1-\alpha)}{\alpha^2(1-\alpha)^2} = \frac{2}{\alpha(1-\alpha)} \Leftrightarrow \quad \theta < \frac{\alpha(1-\alpha)}{2}.
\end{aligned}
$$

E.3 SPC, DQC and notions of aversion to ambiguity

In this appendix, we discuss how conditions SPC and DQC (introduced in Appendix C) are related to notions of aversion to ambiguity that have been discussed in the literature. We start from DQC, as the "decomposability" assumption $V = I \circ u$ is standard in previous work.

DQC

First, we reiterate that DQC is strictly weaker than convexity in utilities, i.e. uncertainty aversion à la Schmeidler (1989). (Similarly, SPC is strictly weaker than convexity in consumption, i.e., preference for diversification.) The examples in Section 2 illustrate these points.

Second, when I is regular, ambiguity aversion in the sense of GM implies DQC. In general, the latter is strictly weaker: see Example 3 in Appendix A. However, DQC implies GM-ambiguity aversion under an additional condition that, for instance, is implied by TIC (Definition 2). Similar results hold for condition SPC and a general V.

Third, Chateauneuf and Tallon (2002) introduce weakenings of convexity in consumption and utility. A preference satisfies *preference for sure EU diversification* if, for all bundles $f_1,\ldots,f_N$ that are mutually indifferent and such that, for suitable weights, their *utility* mixture is a constant $u(x)$, it is the case that $x \succcurlyeq f_n$ for all n. Condition DQC implies a preference for sure EU diversification. The converse holds in certain special cases: for example, it is true for Choquet and VEU preferences, by results in Chateauneuf and Tallon (2002) and Siniscalchi (2009). Whether it holds more generally is an open question.

Chateauneuf and Tallon (2002) also define *preference for sure diversification*: for all bundles $f_1,\ldots,f_N$ that are mutually indifferent and such that, for suitable weights, their *outcome* mixture is a constant x, it is the case that $x \succcurlyeq f_n$ for all n. Condition SPC implies a preference for sure diversification. As above, to what extent the converse implication holds is an open question.

We now provide formal statements and proofs. First, consider a preference $\succcurlyeq$ that admits a representation of the form $V = I \circ u$, such that (I, u) satisfy Assumption 2.

Remark 7 If $\cap_{x>0} C(1_S x) \neq \emptyset$ and DQC holds, then Core $I \neq \emptyset$. In particular, this is the case if V satisfies TIC and DQC holds.

Proof: Let $P \in \cap_{x>0} C(1_S x)$. Consider $a \in u(X)^S$. If $a = 1_S u(0)$, then $I(a) = u(0) = P(a)$ by normalization. If $a \neq 1_S u(0)$, let x be such that $I(a) = u(x)$; this exists by standard arguments. By DQC, for all $Q \in \partial I(1_S u(x))$, $Q(a - 1_S u(x)) \geq 0$. Since $a \neq 1_S u(0)$, by strong monotonicity $I(a) > u(0)$, so $x > 0$. Hence, there is $Q \in \partial I(1_S u(x))$ such that $Q(S) > 0$ and $Q/Q(S) = P$. Therefore, $P(a - 1_S u(x)) \geq 0$, or $P(a) \geq P(1_S u(x)) = u(x) = I(a)$. Since a was arbitrary, $P \in$ Core I.

Under Assumption 2 and DQC, Proposition 12 implies that V satisfies Assumption 1 and SPC. Hence, by Proposition 8, for every $x > 0$, $C(1_S x) = \pi(1_S x)$ and V is nice at $1_S x$. Therefore, $C(1_S x) \neq \emptyset$. Putting these conditions together, since TIC holds, $\bigcap_{x>0} C(1_S x) = \bigcap_{x>0} \pi(1_S x) = \pi(1_S) = C(1_S) \neq \emptyset$. ■

Remark 8 (Preference for sure EU diversification) Suppose that DQC holds. Then, for all bundles $f_1, \ldots, f_N$ and weights $\alpha_1, \ldots, \alpha_N \geq 0$ such that $\sum_i \alpha_i = 1$, if $f_i \sim f_j$ for all i, j, and there exists $x \geq 0$ such that $\sum_i \alpha_i u(f_i(s)) = u(x)$ for all s, then $x \succcurlyeq f_i$.

Proof: Fix f_i, α_i, $i = 1, \ldots, N$, and x as in the statement. Let $y \geq 0$ be such that $f_i \sim y$ for all i; this exists by standard arguments. If $y = 0$, then $f_i = 0$ for all i, so $x = 0$ because u is strictly increasing; then, trivially, $x \sim f_i$ for all i. If instead $y > 0$, DQC implies that $Q(u \circ f_i - 1_S u(y)) \geq 0$ for all $Q \in \partial I(1_S u(y))$. By linearity,

$$Q(1_S u(x) - 1_S u(y)) = Q\left(\sum_i \alpha_i u \circ f_i - 1_S u(y)\right) = Q\left(\sum_i \alpha_i [u \circ f_i - 1_S u(y)]\right) = \sum_i \alpha_i Q(u \circ f_i - 1_S u(y)) \geq 0.$$

Since $Q(S) > 0$ because I is nice at $1_S u(y)$ by assumption, $u(x) \geq u(y)$. hence, $x \succcurlyeq y \sim f_i$ for each i. ■

SPC

Now consider a preference that admits a representation V that is not (necessarily) of the form $I \circ u$. We maintain Assumption 1; in addition, for the first two results, we will assume that V is *normalized*: that is, $V(1_S x) = x$ for all $x \geq 0$. Notice that V can then be interpreted as a certainly-equivalent functional: $V(f) = x$ if and only if $f \sim x$.

The *strict core* of V is the set SCore $V = \{P \in \Delta(S) : \forall f \in \mathbb{R}^S_+$ non-constant, $P(f) > V(f)\}$. For instance, if V is the certainty-equivalent function of an EU preference with strictly positive beliefs P and a strictly concave utility u, then for all non-constant f, $V(f) = u^{-1}P(u \circ f) < u^{-1}(u(P(f))) = P(f)$, so $P \in$ SCore V. If instead the preference is risk-neutral, SCore $V = \emptyset$. Therefore, a non-empty strict core captures a notion of strict risk/ambiguity aversion.

Remark 9 If V is normalized, nice, and regular at certainty (i.e. at every $1_S x$, $x > 0$), and SCore $V \neq \emptyset$, then V satisfies SPC.

The proof mimics that of Corollary 13 part 2, with some additional subtleties.

Proof: Fix $x > 0$ and $f \neq 1_S x$ such that $V(f) \geq V(1_S x)$. If $f = 1_S y$, then by strong monotonicity and the assumption that $f \neq 1_S x$, $y > x$; since $Q(S) > 0$ for all $Q \in \partial V(1_S x)$ by niceness, $Q(1_S y - 1_S x) = Q(S)(y - x) > 0$ for all such Q.

Now suppose that f is non-constant. As in the proof of Corollary 13 part 2, it is enough to show that $V^\ell(1_S x; f - 1_S x) > 0$. Since V is regular, $V^\ell(1_S x; f - 1_S x) = -V^\circ(1_S x; 1_S x - f) = -V'(1_S x; 1_S x - f)$; furthermore, if $V(f) \geq V(1_S x) = x$, by normalization, for any $P \in$ SCore V,

$$
\begin{aligned}
-V^\ell(1_S x; f - 1_S x) = V'(1_S x; 1_S x - f) = \lim_{t \downarrow 0} \frac{V(1_S x + t[1_S x - f]) - V(1_S x)}{t} = \\
= \lim_{t \downarrow 0} \frac{V(1_S x + t[1_S x - f]) - x}{t} \leq \lim_{t \downarrow 0} \frac{P(1_S x + t[1_S x - f]) - x}{t} = \\
= \lim_{t \downarrow 0} \frac{x + tx - tP(f) - x}{t} = x - P(f) \leq V(f) - P(f) < 0,
\end{aligned}
$$

as required. Notice that, while $P(1_S x + t[1_S x - f]) > V(1_S x + t[1_S x - f])$ for positive $t > 0$ because the argument of P and V is non-constant when f is, this may not be true in the limit as $t \downarrow 0$. Thus, the first inequality is weak. However, the last inequality is strict, because f is non-constant and $P \in \text{SCore}\, V$. ■

Remark 10 If V is normalized, $\cap_{x>0} C(1_S x) \neq \emptyset$, and SPC holds, then SCore $V \neq \emptyset$.

Proof: Let $P \in \cap_{x>0} C(1_S x)$ and consider f non-constant. Let $x \sim f$, which exists by standard arguments. Since $x = V(1_S x) = V(f)$, SPC implies that $Q(f - 1_S x) > 0$ for all $Q \in \partial V(1_S x)$. Hence, in particular, $P(f - 1_S x) > 0$, i.e., $P(f) > x = V(f)$. Therefore $P \in \text{SCore}\, V$. ■

Remark 11 (Preference for sure diversification) Suppose that SPC holds. Then, for all bundles $f_1, \ldots, f_N$ and weights $\alpha_1, \ldots, \alpha_N \geq 0$ such that $\sum_i \alpha_i = 1$, if $f_i \sim f_j$ for all i, j, and there exists $x \geq 0$ such that $\sum_i \alpha_i f_i(s) = x$ for all s, then $x \succcurlyeq f_i$ for all i; indeed $x \succ f_i$ for all i unless all f_i's are constant.

Proof: Fix f_i, α_i, $i = 1, \ldots, N$, and x as in the statement. If all f_i's are constant, then the assumptions imply that they must all equal $1_S x$, so the statement holds trivially. By strong monotonicity, this must be the case if $f_i \sim 0$ for all i. Thus, assume that they are not all constant, and that $f_i \succ 0$ for all i. Let $y > 0$ be such that $f_i \sim y$ for all i. SPC implies that $Q(f_i - 1_S y) > 0$ for all $Q \in \partial V(1_S y)$ and all $f_i \neq 1_S y$. By linearity,

$$Q(1_S x - 1_S y) = Q\left(\sum_i \alpha_i f_i - 1_S y\right) = Q\left(\sum_i \alpha_i [f_i - 1_S y]\right) = \sum_i \alpha_i Q(u \circ f_i - 1_S y) > 0.$$

The inequality is strict because $Q(f_i - 1_S y) > 0$ for at least one i. Since $Q(S) > 0$ because V is nice at $1_S y$ by assumption, $x > y$. Hence, $x \succ y \sim f_i$ for each i. ■

E.4 Decomposable representations: Examples

All the following examples feature preferences which have a "decomposable" representation $V = I \circ u$. The first example illustrates that the conditions in Theorem 3 are strictly more general than those discussed in Appendix C. The second example demonstrates that the conditions in Appendix C may be restrictive when combined with specific assumptions about the functional I. The third example shows that the strict quasiconcavity property of Eq. (7) is strictly weaker than SPC.

Example 3 Let $S=\{s_1,s_2\}$. We define the function $V:\mathbb{R}^2_+\to\mathbb{R}$, depicted in Figure 5, in three steps.

First, we define $W_1:\mathbb{R}^2_+\to\mathbb{R}$ by $W(f)=\frac{1}{2}\sqrt{f}_2+\sqrt{4+\frac{1}{4}f_2+2\sqrt{f}_1}-2$. Note that the slope of the indifference curve of W_1 going through the point $1_S x$ (drawn as a dashed black line in Figure 5) equals $-\frac{2}{2+\sqrt{x}}$.

Second, we define $W_2:\mathbb{R}^2\to\mathbb{R}$ by specifying the features of its indifference curves. Fix a constant α (in the picture, $\alpha=1.05$). For any $x>0$, the indifference curve of W_2 going through $1_S(\alpha x)$ is linear, and parallel to the tangent to the indifference curve of W_1 at $1_S x$. Furthermore, $W_2(1_S\alpha x)=\sqrt{x}$ for all $x>0$.[1]

Finally, we let $V(f)=\max(W_1(f),W_2(f))$. By construction, at the point $1_S x$, $W_2(1_S x)<W_1(1_S x)$. Thus, for bundles f near the certainty line, $V(f)=W_1(f)$. However, since the indifference curves of W_2 are flat, whereas those of W_1 bend inward, for bundles f sufficiently far from the 45° line, $W_2(f)>W_1(f)$, so $V(f)=W_2(f)$.

Fix an arbitrary, strictly concave function $u:\mathbb{R}_+\to\mathbb{R}$, and assume that $I:u(X)^2\to\mathbb{R}$ is strictly monotonic and such that $V=I\circ u$. We argue that I is not quasiconcave, and its core is empty; thus, neither condition 1 nor condition 2 in Corollary 13 applies.

Consider two bundles f,g such that $V(f)=V(g)=W_2(f)=W_2(g)$: that is, f and g lie on the same indifference curve for V, in a region where V coincides with W_2 (see Figure 5). Since in that region the indifference curve is linear, $V(\frac{1}{2}f+\frac{1}{2}g)=V(f)$. Therefore, $I(u\circ(\frac{1}{2}f+\frac{1}{2}g))=I(u\circ f)$. However, for every state s, since u is strictly concave and $f(s)\neq g(s)$ because indifference curves are not parallel to the axes, $u(\frac{1}{2}f(s)+\frac{1}{2}g(s))>\frac{1}{2}u(f(s))+\frac{1}{2}u(g(s))$. Since I is

[1] The details are as follows. Since the indifference curve of W_2 is linear, it consists of points $f=(f_1,f_2)$ such that $f_2=mf_1+q$; for $f_1=f_2=\alpha x$, by assumption the slope is $-\frac{2}{2+\sqrt{x}}$, so $q=\frac{4+\sqrt{x}}{2+\sqrt{x}}\alpha x$. Hence the indifference curve of W_2 going through $1_S\alpha x$ has equation $f_2=-\frac{2}{2+\sqrt{x}}f_1+\frac{4+\sqrt{x}}{2+\sqrt{x}}\alpha x$. Since any $f\in\mathbb{R}^2_+$ lies on a *unique* indifference curve, and each indifference curve is parameterized by x, the value of $W_2(f)$ for an arbitrary $f\in\mathbb{R}^2_+$ can be computed as follows. First, find the unique x such that f satisfied the linear equation parameterized by x; this can be done numerically, using any one-dimensional search algorithm. Second, note that then f lies on the same indifference curve as $1_S\alpha x$, so by assumption $W_2(f)=W_2(1_S\alpha x)=\sqrt{x}$.

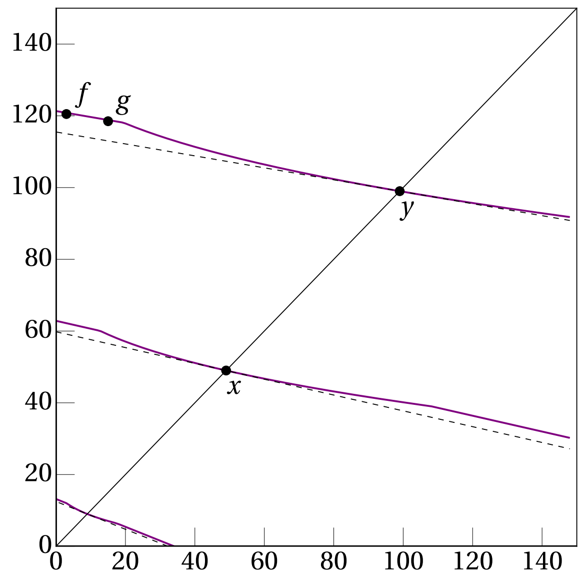

Figure 5: A non-convex preference with an empty core that nevertheless satisfies SPC

strictly monotonic, $I(u\circ(\frac{1}{2}f+\frac{1}{2}g)) > I(\frac{1}{2}u\circ f+\frac{1}{2}u\circ g)$. Conclude that $I(u\circ f) > I(\frac{1}{2}u\circ f+\frac{1}{2}u\circ g)$: but then, I is not quasiconcave.

Next, Proposition 16 in Section 5 and Proposition 19 in Appendix D.5 imply that Core $I \subseteq \bigcap_{x>0} C(1_S x)$.[2] But, since $V = W_1$ near every constant bundle $1_S x$, and W_1 is smooth, $C(1_S x) = \{P_x\}$, where P_x identifies the line supporting the upper contour set of W_1 at $1_S x$, which therefore has slope $-\frac{2}{2+\sqrt{x}}$. Since $x \neq y$ implies $P_x \neq P_y$, Core $I = \emptyset$.

Finally, we show that Condition SPC is satisfied. Suppose that $V(f) \geq V(1_S x) = W_1(1_S x)$. If $V(f) = W_1(f)$, then $W_1(f) \geq W_1(1_S x)$ implies that $\nabla V(1_S x)\cdot(f-1_S x) = \nabla W_1(1_S x)\cdot(f-1_S x) > 0$ because W_1 is strictly quasiconcave. If instead $V(f) = W_2(f)$, let $y \geq 0$ be such that $V(f) = W_2(f) = V(1_S y) = W_1(1_S y)$; this is the case for the points labelled f, x, y in Figure 5. Then $y \geq x$, and f lies on an indifference curve for W_2 that is parallel to, but *higher* than the indifference curve for W_1 through $1_S y$, and hence also *higher* than the indifference curve for W_1 through $1_S x$. But this means that, again, $\partial V(1_S x)\cdot(f-1_S x) = \partial W_1(1_S x)\cdot(f-1_S x) > 0$.

[2]For every $x > 0$, Proposition 19 implies that Core $I \subseteq \pi^s(1_S x)$; by parts 2 and 1 of Proposition 16, $\pi^s(1_S x) \subseteq \pi(1_S x) \subseteq C(1_S x)$ [note that $\partial W_1(1_S x) \neq 0_S$, so V is nice at $1_S x$]; the claim follows.

Example 4 (Invariant Biseparable preferences) A preference is *invariant biseparable* (Ghirardato, Maccheroni, and Marinacci, 2004) if its representation (I, u) is such that I is positively homogeneous and constant-additive on its domain: for all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$, $I(\alpha + \beta a) = \alpha + \beta I(a)$ (this implies that I is Lipschitz with constant 1 and normalized). We now show that MEU preferences are the only invariant biseparable preferences for which condition DQC in Appendix C holds.

Recall from Ghirardato et al. (2004) that, for an invariant biseparable preference represented by (I, u), the functional I admits a unique extension to all of $\mathbb{R}^S$, and the Clarke subdifferential at zero, i.e., $\partial I(0_S)$, consists of probability measures and coincides with $\partial I(1_S u(x))$ for all $x > 0$. Hence, I is nice at $1_S u(x)$ for every $x > 0$.

If preferences are MEU, then I is concave, so DQC holds by Corollary 13. Conversely, assume that DQC holds. Let $D = \partial I(0_S) = \partial I(1_S) \subseteq \Delta(S)$. Then, Proposition 19 and Corollary 20 in the Appendix imply that $D = \text{Core } I$. But by Proposition 16 in Ghirardato et al. (2004), $D = \text{Core } I$ if and only if I is concave, in which case the preference is MEU.

Thus, for invariant biseparable preferences, the sufficient condition for SPC provided by Proposition 19 only holds for the special case of MEU. □

Example 5 (Strict pseudoconcavity vs. strict quasiconcavity at a point) Suppose $S = \{s_1, s_2, s_3\}$; write the generic vector $v \in \mathbb{R}^S$ as $v = (v_1, v_2, v_3)$. Fix a strictly concave, strictly increasing utility function $u : \mathbb{R}_+ \to \mathbb{R}$, a number $\epsilon \in (0, \frac{1}{2})$, let $p^1 = (1-\epsilon, \frac{\epsilon}{2}, \frac{\epsilon}{2})$, $p^2 = (\frac{\epsilon}{2}, 1-\epsilon, \frac{\epsilon}{2})$, $p^{31} = (\frac{2}{3}\epsilon, \frac{1}{3}\epsilon, 1-\epsilon)$ and $p^{32} = (\frac{1}{3}\epsilon, \frac{2}{3}\epsilon, 1-\epsilon)$, and finally define $I : u(X)^S \to \mathbb{R}$

$$I(a) = \min\Big(p^1 \cdot a, p^2 \cdot a, \max\big(p^{31} \cdot a, p^{32} \cdot a\big)\Big)$$

for every $a \in u(X)^S$.

Since all the probabilities defined above are strictly positive, I is strictly increasing. It is also invariant biseparable. Furthermore, we can normalize u so that $u(0) = 0$ and $u(1) = 1$. By Proposition 12, $V = I \circ u$ satisfies Assumption 1.

We show that V is strictly quasiconcave at every $1_S x$, $x > 0$ (i.e., Eq. (7) holds at certainty). Suppose that $V(g) \geq V(1_S x)$ and $g \neq 1_S x$. Then $u \circ g \neq 1_S u(x)$; therefore, since u is strictly concave, for every $\lambda \in (0, 1)$, $u(\lambda g(s) + (1-\lambda)x) \geq \lambda u(g(s)) + (1-\lambda)u(x)$ for every state s, with at least one strict inequality. Since I is strictly increasing, $I(u \circ (\lambda g + (1-\lambda)x)) > I(\lambda u \circ g + (1-

$\lambda)u(x))$; and since I is constant-linear, $I(\lambda u \circ g + (1-\lambda)u(x)) = \lambda I(u \circ g) + (1-\lambda)u(x) \geq u(x)$. This proves the claim.

Since I is invariant biseparable but not concave,[3] Example 4 implies that it does not satisfy DQC. We now show that, in addition, V fails SPC. Let $g = (1, \delta, 0)$, where $u(\delta) = 1-\epsilon$. Then we have $p^1 \cdot u \circ g = p^1 \cdot (1, 1-\epsilon, 0) = (1-\epsilon) + \frac{\epsilon(1-\epsilon)}{2}$, $p^2 \cdot u \circ g = \frac{\epsilon}{2} + (1-\epsilon)^2$, $p^{31} \cdot u \circ g = \frac{2}{3}\epsilon + \frac{\epsilon(1-\epsilon)}{3}$ and $p^{32} \cdot u \circ g = \frac{1}{3}\epsilon + \frac{2\epsilon(1-\epsilon)}{3}$. Thus, for ϵ small, $V(g) = p^{31} \cdot u \circ g > p^{32} u \circ g$. Furthermore, $p^{32} u \circ g \in (0,1)$, so there is $x \in \mathbb{R}_{++}$ with $u(x) = V(1_S x) = p^{32} u \circ g$. We then have $V(g) > V(1_S x)$. However, $p^{32} \in C(1_S x)$ and

$$p^{32} \cdot g = \frac{1}{3}\epsilon + \frac{2}{3}\delta\epsilon < \frac{1}{3}\epsilon + \frac{2}{3}(1-\epsilon)\epsilon = p^{32} \cdot u \circ g = x = p^{32} \cdot 1_S x;$$

the inequality holds because, by strict concavity of u,

$$u(\delta) = 1-\epsilon = (1-\epsilon)\cdot 1 + \epsilon \cdot 0 = (1-\epsilon)u(1) + \epsilon u(0) < u((1-\epsilon)\cdot 1 + \epsilon \cdot 0) = u(1-\epsilon),$$

and since u is strictly increasing, $\delta < 1-\epsilon$.

With reference to Example 4, the preference described here is invariant biseparable and satisifies strict *quasi*concavity at every $1_S x$, $x > 0$ (cf. Eq. (7)), but is not MEU. It is an open question whether strict *pseudo*concavity at certainty may hold for invariant biseparable preferences that are not MEU. □

E.5 Examples: Convex preferences

We conclude with two examples with convex preferences. Example 6 shows that, even when all preferences are strictly convex, a non-empty intersection of the supporting-probabilties sets $\pi_i(\cdot)$ at *every* full-insurance allocation is necessary for risk sharing. Example 7 instead illustrates how risk sharing may obtain when TIC fails—that is, when Theorem 3 applies but 4 does not.

Example 6 Let $S = \{s_1, s_2\}$. Agent 1's preferences are represented by the utility function $u_1(x) = x^{0.6}$ and the differentiable, quasiconcave, but not concave functional

$$I(a) = \frac{1}{2}a_2 + \sqrt{4 + \frac{1}{4}a_2^2 + 2a_1} - 2.$$

[3] Consider $a = (2,1,0)$ and $b = (1,2,0)$ with linear utility. For ϵ small, $I(a) = I(b) = \frac{5}{3}\epsilon$, but $I(\frac{1}{2}a + \frac{1}{2}b) = \frac{3}{2}\epsilon$.

Agent 2 has EU preferences, with probability P and utility $u_2(x) = x^{0.8}$. Figure 6 shows indifference curves for these preferences, drawn as solid blue and red lines respectively. Agent 1's and 2's indifference curves are tangent at the allocation $(1_S x^l, 1_S(\bar{x} - x^l))$; their common slope there equals the slope of the two parallel, straight purple lines. (Thus, this slope identifies P.)

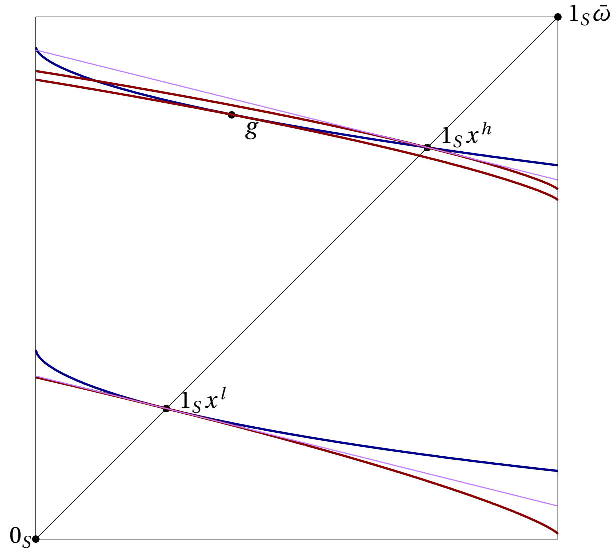

Figure 6: A convex preference with empty core.

The figure shows that the slope of 1's indifference curves at $1_S x^l$ and $1_S x^h$ is different; indeed, it may be verified that the slope of the indifference curve of I at $1_S\gamma$ is $-\frac{2}{\gamma+2}$ for every $\gamma > u_1(0)$; this is non-zero and strictly decreasing in γ. Hence, I is nice at certainty. Furthermore, since I is quasiconcave and u_1 is strictly concave, $V_1 = I \circ u_1$ satisfies SPC by Corollary 13, and therefore by Proposition 16, $\pi_1^s(1_S x) = \pi_1(1_S x) = C_1(1_S x)$ for all $x > 0$. In particular, $\pi_1(1_S x)$ is a singleton set. On the other hand, since agent 2's preferences are consistent with EU, $\pi_2^s(1_S x) = \pi_2(1_S x) = C_2(1_S x) = \{P\}$.

From a decision-theoretic perspective, we observe that agent 1's preference is uncertainty-averse in the sense of Schmeidler, 1989, because I is quasiconcave; however, it is not GM-ambiguity-averse.[4] To see this, note that, by Corollary 13 part 1, together with Corollary 20 in Appendix C, the core of I must be contained in the sets $\pi_1(1_S x)$ for all $x > 0$, but as noted above

[4]Another example of a preference which is convex but not GM-ambiguity-averse can be found in Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011).

these sets are all singleton (hence, non-empty) and different for different x, so Core $I = \emptyset$. For the same reason, TIC fails.

Turn now to risk sharing. The assumptions of Theorem 3 hold. The purple line going through x^l corresponds to a shared supporting probability: that is, $\pi_1(1_S x^l) \cap \pi_2(1_S(\bar{x} - x^l)) \neq \emptyset$. However, the purple line going through x^h is tangent to agent 2's indifference curve, but does *not* support agent 1's indifference curve: therefore, $\pi_1(1_S x^h)$ does not intersect $\pi_2(1_S(\bar{x} - x^h))$. Thus, condition (iv) in Theorem 3 is violated. Correspondingly, conditions (ii) and (iii) also fail: the allocation $(g, 1_S \bar{x} - g)$ is Pareto-efficient, but does not provide full insurance, whereas the interior, full-insurance allocation $(1_S x^h, 1_S(\bar{x} - x^h))$ is not Pareto-efficient.

Finally, note that the interior, full-insurance allocation $(1_S x^l, 1_S(\bar{x} - x^\ell))$ is Pareto-efficient; thus, in this economy, condition (i) in Theorem 1 holds. However, as just noted, conditions (ii)-(iv) in Theorem 3 do not hold. Thus, this example demonstrates that condition (i) cannot be included in the statement of Theorem 3. □

Example 7 Modify Example 6 by assuming that agent 2's preferences are MEU, with priors $D \equiv \{P \in \Delta(S) : P(\{s\}) \in [0.4, 0.6]\}$; furthermore, assume that both agents have utility $u_i(x) = \sqrt{x}$. Refer to Figure 7.

Both preferences are strictly convex and admit a decomposable representation $V_i = I_i \circ u_i$, $i = 1, 2$. Observe that Assumption 2 holds; furthermore, the functionals I_i are both quasiconcave, so both preferences satisfy SPC by Corollary 13. In addition, agent 2's preferences satisfy TIC: this follows because $\partial I_2(1_S x) = C_2(1_S x) = D$ for every $x > 0$.[5]

From Corollary 13 and Corollary 20 in Appendix C, $\pi_2(1_S x) = \pi_2^c(1_S x) = C_2(1_S x) = D$ for every $x > 0$. From Proposition 16, since each V_i satisfies SPC, $\pi_i(1_S x) = \pi_1^s(1_S x)$ for $i = 1, 2$ and $x > 0$. Therefore, for every $x > 0$, $\pi_1(1_S x) \cap \pi_2(1_S x) \neq \emptyset$. Proposition 18 and Theorem 3 apply, and indeed the set of Pareto-efficient and full-insurance allocations coincide.

Since agent 1's preferences do not satisfy TIC, Theorem 4 does not apply. Indeed, neither does RSS's original risk-sharing result (Proposition 9 in their paper): while both agents' preferences are convex (and indeed 2's preferences have a concave representation), agent 1's

[5] By Proposition 19, $C_2(1_S x) = C_2^u(1_S x)$, and $C_2^u(1_S x) = D$ because I_2 is the MEU preference functional, with priors D (Ghirardato et al., 2004).

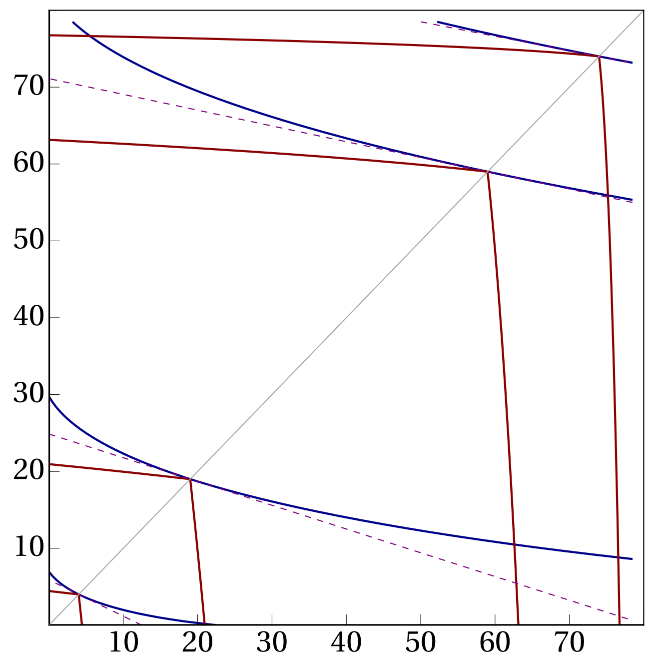

Figure 7: Risk sharing with non-constant supporting probabilities at certainty.

supporting probabilities are not constant at certainty. □

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