Efficient Cooperation in Simple Strategies

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Abstract

We study chip-strategy equilibria in two-player repeated games. Intuitively, in these equilibria players exchange favors by taking individually suboptimal actions if these actions create a "gain" for the opponent larger than the player's "loss" from taking it. In exchange, the player who provides a favor implicitly obtains from the opponent a chip that entitles the player to receiving this kind of favor at some future date. Players are initially endowed with a number of chips, and a player who runs out of chips is no longer entitled to receive any favors until she provides a favor to the opponent, in which case she receives one chip back.

We show that such simple chip strategies approximate efficient outcomes in a class of repeated games with incomplete information, when discounting vanishes. This class includes many important applications, studied in numerous previous papers, such as the discrete-time favor exchange model of Möbius (2001), repeated auctions, and the repeated Spulber's duopoly in Athey and Bagwell (2001), among others.

1 Introduction

Everyday cooperation relies on simple principles, such as an exchange of favors. People easily forgive others with whom they often interact for minor offenses or improper behavior. Employees typically agree when fellow employees occasionally ask them for help, or to replace them in performing some duty. Examples abound in various spheres of human relations. This kind of behavior is most likely driven by the expectation of reciprocity when the roles happen to be reversed. However, long sequences of favors performed in one direction, without any reciprocal behavior, are unlikely to be observed. Favor providers would finally say that they have had enough.

The literature on repeated games or, more generally, dynamic games is very successful in explaining cooperation. Yet this literature emphasizes equilibrium payoffs more than equilibrium behavior. Cooperation is quite often (i) supported by trigger strategies, which penalize any misbehavior by breaking cooperation, or (ii) attained in strategies constructed by self-generation techniques, which are powerful for characterizing payoffs, but typically not useful for characterizing behavior.

In this paper, we formally study a simple form of favor exchange, which is often called *chip strategies*. Chip strategies were introduced in the context of a (two-player) favor-exchange model by Möbius (2001). Intuitively, according to these strategies a player takes an individually suboptimal action if that action creates a "gain" for the opponent larger than the player's "loss" from taking it. In exchange, the player implicitly obtains from the opponent a chip that entitles the player to receive this kind of favor at some future date. Players are initially endowed with a number of chips, and a player who runs out of chips is no longer entitled to receive any favors, until she provides a favor to her opponent, in which case the player receives one chip back.

We show that *simple chip strategies*, in which one favor is exchanged for one chip, are capable of approximating efficient outcomes (for the discount factor tending to 1) in a class of games which has a large array of applications. This class of games includes several models studied extensively in the existing literature, such as the favor exchange model, repeated auctions, or a repeated duopoly.[1]

Any attempt to explore behavior in repeated or dynamic interactions encounters numerous difficulties. One of them is that "natural" patterns of behavior which resemble what is observed in some real-life interactions are unlikely to be widely observed (for a large range of games, monitoring or information structures). Instead, they are observed

[1] In addition, we generalize the existing results obtained in specific applications by allowing types to evolve over time according to a Markov process. Markov types have previously been studied only in the context of a repeated oligopoly.

only in some specific settings.[2]

We also encounter these difficulties. In particular, our result requires strong assumptions: the stage game must be symmetric, it must be played by two players, and each player's private information must be captured by two possible types. We argue that these are essential limitations on the possibility of approximating efficient outcomes in chip strategies. We show that if a stage game is asymmetric, simple chip strategies typically cannot approximate efficient outcomes. One needs to augment them with a public randomization device, which allows players to exchange one favor for receiving some chips only with some probability, or in other words, to introduce divisible chips. This is what we call *random chip strategies*. We show that if players have more than two types, even these random chip strategies may not approximate efficient outcomes.

In the case of more than two players, an additional difficulty arises if a favor is provided by more than one player, and more than one player benefits from this favor. In this case, it must be decided who should issue a chip (or what fraction of it) and who should obtain it. This implies that if efficiency is to be attained, it must be attained in more complicated strategies. We have no fully satisfying extension of chip strategies to more general games. However, in a companion paper, Olszewski and Safronov (2015), we suggest a more complicated version of chip strategies, constructed by imitating the d'Aspremont and Gerard-Varet (1979) mechanism, which attain the efficient outcomes for a large class of games with any number of types and players.

We view our main contribution as providing a positive model of playing some repeated games with incomplete information, which may evolve over time according to a Markov process. Our result shows that chip strategies approximate efficient cooperation in several settings studied in the existing literature. We provide numerous applications, including (i) to the favor-exchange model studied Möbius (2001) (more precisely, to a discrete version of this model studied by Abdukadiroglu and Bagwell (2012, 2013)); (ii) to the repeated Spulber's (1995) duopoly studied by Athey and Bagwell (2001) and several other authors; and to repeated auctions studied by Skrzypacz and Hopenhayn (2004) and others. We show

[2]Another issue is that different observers view different patterns of behavior to be more commonly observed, in real-life interactions.

that in these models the efficient outcome can be approximated in simple chip strategies, when the discount factor converges to 1; moreover, the proof of these results are engagingly simple, and potentially applicable to other settings, even ones not directly covered by our general results.

The rest of the paper is organized as follows. In Section 2, we present two applications: to a favor-exchange model and to repeated auctions. These applications are instructive also from the technical perspective. In the former application, we explain the reasons for attaining efficiency in chip-strategy equilibria in the simplest possible manner. The proof for the latter application is in turn very close to the proof of our general result; the main difference comes from the fact that we allow for Markov types in the general result, which requires some arguments to be more involved. Section 3 contains the general result, and Section 4 provides some additional applications, including to the repeated duopoly model of Athey and Bagwell (2001). Sections 5-7 contain a discussion of chip strategies, and a discussion of possible extensions of the result from Section 3. Technically demanding proofs are relegated to appendices.

1.1 Related literature

To date, chip strategies have appeared in the existing literature only in the context of specific applications (which we will discuss shortly). However, no systematic studies of chip strategies have been carried out prior to this paper.

Möbius (2001) analyzed a model of voluntary favor exchange between two players. In his model, favor opportunities arrive according to a Poisson process, and the benefit of receiving a favor exceeds the cost of providing it. In the chip strategies studied by Möbius (which are somewhat different and less efficient than those studied in our paper), cooperation breaks down when a player issues a certain number of chips. Hauser and Hopenhayn (2008) suggest two improvements to chip strategies that enhance the efficiency of equilibria: exchanging favors and chips at different rates (i.e., not one to one), and appreciation and depreciation of chips. Solving the model numerically, they demonstrate that for a large set of parameter values, the efficiency gains are quite large. As already mentioned earlier, Abdukadiroglu and Bagwell (2012, 2013) studied a discrete-time version of Möbius' model. Their 2012 paper is closely related to our paper, because they analyze chip strategies in the same form as we do here. They consider a fixed discount factor, identify the optimal number of chips, and compare this optimal chip mechanism with a more sophisticated favor-exchange relationship in which the size of a favor owed may decline over time. For any given discount factor, the equilibria in chip strategies obviously cannot be fully efficient, because incentive compatibility imposes a limit on the number of chips that can be used. However, none of these papers shows (explicitly or implicitly) that any kind of chip strategies attain efficient outcomes when the discount factor converges to 1. With no restriction on strategies, the possibility of attaining efficient cooperation follows from Fudenberg, Levine, and Maskin (1994).

Repeated Spulber's duopolies (more generally, oligopolies) with i.i.d. types were studied in Athey and Bagwell (2001, 2008), Athey, Bagwell, and Sanchirico (2004), Hörner and Jamison (2007), and - with more general Markov types - by Escobar and Toikka (2010). Even though the strategies in Athey and Bagwell (2008) resemble our one-chip strategy profiles, and some elements of chip strategies appear in Hörner and Jamison (2007), it remains the case that the primary focus of these papers is not on chip strategies per se. Compared to the strategies used by these authors (even for i.i.d. types), chip strategies are remarkably simple for showing that efficient outcomes can be approximated when the discount factor converges to 1.

Skrzypacz and Hopenhayn (2004), Blume and Heidhues (2008), and Rachmilevich (2013) study collusion in repeated auctions without communication, and show that players can obtain a higher equilibrium total payoff than in (i) the repetition of stage-game Nash equilibrium, or (ii) the bid rotation scheme in which players alternate between being the winner and the loser. These strategy profiles differ from (and are less efficient than) chip strategies, because the winners are determined exogenously. The strategies in Skrzypacz and Hopenhayn, and Blume and Heidhues are defined only implicitly, and Rachmilevich studies a version of one-chip equilibria in which the current loser decides who will be the winner in the following period. Allowing for mediated communication, Aoyagi (2007) shows that efficiency can be attained for a large class of repeated-auction settings. He obtains these results by modifying self-generation techniques.[3] The strategies that attain efficient payoffs resemble chip strategies in two-player auctions; however, players providing favors are not compensated by obtaining chips but instead are compensated directly by an increase in continuation payoffs, which makes the strategies less explicit.[4]

2 Applications, part I

2.1 Discrete model of favor exchange

In this section, we study a model of Abdulkadiroglu and Bagwell (2012), which in turn is a discrete version of the model of favor exchange in Möbius (2001). In the stage game of their model, either player 1 is given an income of \$1 or player 2 is given an income of \$1, or neither player is given any income. The former two events occur with probability $p \in (0, 1/2)$ each, and the latter event occurs with probability $1 - 2p$. Each player is privately informed as to whether or not she receives income. Thus, if a player does not receive income, then she does not observe either whether the opponent received any income. If a player receives income, then the player may send the income to the other player. The transferred income is worth $\gamma > 1$ to the receiver. So, making it value-enhancing.

This game is played repeatedly, states are *i.i.d.*, and players have a common discount factor δ. The payoffs are normalized by the factor of $1 - \delta$. Players cannot store income, that is, income must be either transferred or consumed in the period it is received. The (ex ante) *efficient (total) payoff*, that is, the maximum of the sum of the players' payoffs, is

$$v = 2p\gamma.$$

That is, income should be transferred whenever received.

[3]He could not apply the results in Fudenberg, Levine, and Maskin (1994) directly, because, unlike other cited papers, he allows for players' payoffs to depend on other players' signals.

[4]Similarly, the strategies used in auctions with more than two players, and the strategies studied in Aoyagi (2003), share some features with the strategies used in Olszewski and Safronov (2015) to show that efficient outcomes are attainable in a larger class of games with any number of players and players' types.

2.1.1 Description of efficient chip strategies

Consider the following strategies: at the beginning of each period, each player i holds $k_i \in \{0, ..., 2n\}$ chips, where $k_1 + k_2 = 2n$. If player i obtains an income of \$1, and $k_i < 2n$, then player i gives the income to player j, and j gives i (implicitly) one chip in return. If $k_i = 2n$, i.e., when i already holds all the chips, then i consumes the \$1 herself. No chip is given in this case. At the beginning of period 1, each player has n chips.

We obtain the following result, proved in the next two sections:

Theorem 1 *For every $\lambda > 0$, there is a $\underline{\delta} < 1$ such that for every $\delta > \underline{\delta}$, there is a chip-strategy equilibrium of the repeated game in which the players' discount factor is δ, such that the ex ante payoff of each player in this equilibrium exceeds $v/2 - \lambda$.*

2.1.2 Continuation payoffs

Assuming that both players play the prescribed chip strategies, denote the continuation payoff of player 1 with k chips by V_k. By the symmetry of our model, the continuation payoff of player 2 can be expressed analogously. These continuation payoffs are computed before the players learn about their income in the current period. For $k \in \{1, ..., 2n-1\}$, we have:

$$V_k = p\left\{(1-\delta)\gamma + \delta V_{k-1}\right\} + p\delta V_{k+1} + (1-2p)\delta V_k. \tag{1}$$

Indeed, the first component of the right-hand side corresponds to the payoff contingent on player 2 receiving an income of \$1 in the current period; the remaining two components correspond to player 1 receiving an income of \$1 and no player receiving any income, respectively.

For $k = 0$ and $2n$, we have:

$$V_0 = p\delta V_0 + p\delta V_1 + (1-2p)\delta V_0, \tag{2}$$

$$V_{2n} = p\{(1-\delta)\gamma + \delta V_{2n-1}\} + p\{(1-\delta) + \delta V_{2n}\} + (1-2p)\delta V_{2n}. \tag{3}$$

2.1.3 Payoff Efficiency and Incentive Constraints

We can now demonstrate the efficiency of the prescribed strategies:

Proposition 1 *For any given n and $\lambda > 0$, there is an $\overline{\delta} > 0$ such that for every $\delta < \overline{\delta}$, we have:*

$$V_k > \frac{2n-1}{2n+1} p\gamma - \lambda$$

for all $k = 0, ..., 2n$.

Proof. The strategies induce a stochastic Markov chain over states $k = 0, ..., 2n$. By the Ergodic Theorem (see, for example, Chapter 1, §12, Theorem 1 in Shiryaev, 1996) there exists a probability distribution over states $\{\pi_k : k = 0, ..., 2n\}$ such that the probability of being in state k after a sufficiently large number of periods is arbitrarily close to π_k, independent of the initial state.[5] This probability distribution is an eigenvector corresponding to eigenvalue 1 of the transition matrix

$$\begin{bmatrix} 1-p & p & 0\,.\,. & . & 0 \\ p & 1-2p & p\,. & & . \\ 0 & . & ... & & . \\ . & . & ... & . & . \\ . & & ... & . & 0 \\ . & & .\,p & 1-2p & p \\ 0 & . & .\,.\,0 & p & 1-p \end{bmatrix},$$

in which the entry in row i and column j is equal to the probability of transiting from state i to state j. Thus, it is easy to verify that the eigenvector corresponding to eigenvalue 1 must have coordinates equal to $1/(2n+1)$.

Since the expected payoff of each player in any state other than 0 and $2n$ is $p\gamma$, then it follows that, when δ is sufficiently close to 1, the player's continuation payoff is $p\gamma$ with probability arbitrarily close to $(2n-1)/(2n+1)$, which is the limit occupation probability of states other than 0 and $2n$. ■

It remains to show the incentive compatibility of the prescribed strategies. A player can only deviate when she has income, by consuming the income herself rather than transferring it to the opponent. The gain of this deviation, compared to playing as prescribed, is 1.

[5] One can easily verify that the assumptions of the Ergodic Theorem are satisfied; indeed, each state is reached from any other state in $2n$ periods with positive probability.

In turn, the gain from playing the prescribed strategy, compared to deviating, is that the player will have one more chip in the next period. We will now prove that this gain is no smaller than 1.

Proposition 2 *For every n, there is a $\underline{\delta} < 1$ such that for every $\delta > \underline{\delta}$, we have that $\Delta_k := V_k - V_{k-1} > 1$ for all $k = 1, ..., 2n$.*

Proof. From equations (1)-(3), by subtracting the equation for V_{k-1} from the equation for V_k, we obtain:

$$\Delta_k = p\delta\Delta_{k-1} + p\delta\Delta_{k+1} + (1-2p)\delta\Delta_k$$

for $k = 2, ..., 2n-1$;

$$\Delta_1 = p(1-\delta)\gamma + p\delta\Delta_2 + (1-2p)\delta\Delta_1;$$

and

$$\Delta_{2n} = p(1-\delta) + p\delta\Delta_{2n-1} + (1-2p)\delta\Delta_{2n}.$$

For $\delta = 1$, this system of linear equations is satisfied when all Δ's are equal to 0. For $\delta < 1$, the system is harder to solve, so we will evaluate Δ's in approximation by referring to the Implicit Function Theorem. By this theorem, one can differentiate the equations for Δ's with respect to δ, plug in $\delta = 1$ and $\Delta_k = 0$ for all k, and obtain a system of equations for the derivatives of Δ's. This implies that if we replace each Δ_k by its derivative $\partial\Delta_k/\partial\delta$, then our system of equations must be satisfied with $\delta = 1$, and with the free terms $p(1-\delta)\gamma$ and $p(1-\delta)$ replaced with their derivatives $-p\gamma$ and $-p$, respectively. In the matrix notation and after we divide by p, this new system of linear equations can be expressed as

$$\begin{bmatrix} 2 & -1 & 0 & . & . & . & 0 \\ -1 & 2 & -1 & . & & & . \\ 0 & . & . & . & . & & . \\ . & . & . & . & . & . & . \\ . & & . & . & . & . & 0 \\ . & & & . & -1 & 2 & -1 \\ 0 & . & . & . & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \partial\Delta_{2n}/\partial\delta \\ . \\ \\ . \\ \\ . \\ \partial\Delta_1/\partial\delta \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ . \\ . \\ . \\ 0 \\ -\gamma \end{bmatrix}.$$

This system of linear equations can be easily solved by the Gauss-Jordan elimination method; the unique solution is given by

$$\frac{\partial \Delta_k}{\partial \delta} = -\frac{(2n-k+1)\gamma + k}{2n+1} < -1$$

for all k. This means that $\Delta_k > 1$ for $k = 1, ..., 2n$ and δ close to 1. ■

Two comments are essential. First, the matrix of this system of linear equations is nonsingular, which is equivalent to the uniqueness of our solution. This validates the use of the Implicit Function Theorem. Second, notice that we cannot use the Implicit Function Theorem to approximate V's directly, because the matrix of the system of equations (1)-(3) linearized at $\delta = 1$ would have a determinant equal to zero.

2.2 Repeated auctions

In this section, we study the model of repeated auctions. In every period, two players participate in a first-price auction;[6] ties are resolved by a fifty-fifty lottery. At the end of each period the identity of the winner, but not the bids, is revealed. Each player receives a private signal about the object. These signals can take one of two values, H or L; they are i.i.d. over time, but may be correlated across players. We restrict attention to symmetric signal structures, that is, the probability distribution over type profiles is exhibited in the following table:

$$\text{Player 2's signal} \begin{bmatrix} H & q & 2p \\ L & 1-2p-2q & q \\ & L & H \end{bmatrix}, \\ \text{Player 1's signal}$$

where $p, q \geq 0$ and $p + q < 1/2$.

A player's valuation of the object is a function of both signals: that of the player herself and that of the player's opponent. We restrict attention to symmetric valuations. That

[6] We assume this particular auction format only for concreteness. It is easy to check that all our arguments apply to any auction in which (a) the player who makes a higher bid wins the object; (b) the payment of a player who bids 0 is 0; and (c) the payment is continuous at 0, that is, the payment of a player who bids close to 0 must also be close to 0.

is, players have the same valuation function v. The valuations are strictly positive, and weakly increasing in each signal, and each player's valuation increases in the player's own signal by no less than in the signal of the player's opponent. That is, $v(H,H) \geq v(H,L) \geq v(L,H) \geq v(L,L) > 0$. Players have a common discount factor δ. We allow for sending cheap-talk messages at the beginning of each period.

The efficient winner of the object is depicted in the following table:

$$\begin{array}{cc} \text{Player 2's signal} & \left[\begin{array}{ccc} H & \text{player 2} & \text{any player} \\ L & \text{any player} & \text{player 1} \\ & L & H \end{array}\right] . \\ & \text{Player 1's signal} \end{array}$$

2.2.1 Description of efficient chip strategies

At the beginning of each period, each player i holds $k_i \in \{0, ..., 2n\}$ chips, where $k_1 + k_2 = 2n$. At the beginning of period 1, each player has n chips. If $0 < k_1 < 2n$, players report their signals truthfully in the cheap-talk communication. If both players report the same signal, they determine by a fifty-fifty lottery who will be the winner in the current period.[7] At the bidding stage: (i) a player bids ρ if her signal is H and the opponent's signal is L; (b) she also bids ρ when players have the same signals, and the player was determined to be the winner in the lottery; (c) if her signal is L and the opponent's signal is H, the player bids 0; and (d) she also bids 0 when players have the same signals, and the opponent was determined to be the winner of the lottery. These actions generate the desired efficient outcome, and before players submit their bids, they know who is supposed to be the winner in the current period. A player with no chips bids 0, and a player with all $2n$ chips bids ρ, regardless of signals. If $0 < k_1, k_2 < 2n$, the winner gives the loser a chip; and when $k_i = 2n$ and $k_j = 0$, player i gives player j a chip.

If a player who was not supposed to win in the current period happens to win (which cannot ever happen if players play the prescribed strategies), players switch to playing a symmetric stage-game Bayesian Nash equilibrium from the following period. It is a standard exercise to show that such an equilibrium exists, and the bids in such an equilibrium

[7] As we argue later, this lottery can be performed by means of cheap-talk messages.

are never lower than $\min\{E(v(\cdot \mid L), E(v(\cdot \mid H)\} > 0$, where $Ev(\cdot \mid t)$ is the expected value of the object for a player, contingent on t being the player's signal. Therefore, in the stage-game equilibrium, each player's payoff is strictly lower than one half of the efficient payoff. Thus, in states other than 0 and $2n$ players obtain a higher expected payoff by playing the strategies prescribed than by playing the stage-game equilibrium. The exact form of the stage-game equilibrium will be inessential for our purposes.

We obtain the following result, proved in the next two sections:

Theorem 2 *For every $\lambda > 0$, there is a $\underline{\delta} < 1$ such that for every $\delta > \underline{\delta}$, there is a chip-strategy equilibrium of the repeated auction in which the players' discount factor is δ such that the ex ante expected total payoff in this equilibrium cannot be lower than the efficient total payoff by more than λ.*

Note that our result generalizes a result in Aoyagi (2007; see the discussion after Corollary 8). In the symmetric two-player, two-signal case Aoyagi proves that the efficient outcome can be approximated only under somewhat stronger assumptions (see his Assumption 1). More importantly, Aoyagi is not concerned with the form of strategies that approximate efficient payoffs. We note, however, that Aoyagi also obtains results for other IR and feasible payoff vectors, or other Pareto-efficient payoff vectors, as well as results for more than two players and more than two signals.

2.2.2 Continuation payoffs

Assuming that both players play the prescribed strategies, denote by V_k the continuation payoff of player 1 who currently has k chips. We will focus on player 1; the analysis of player 2's continuation payoffs is analogous by symmetry. These continuation payoffs are computed before the players learn about their current signals. For $k \in \{1, ..., 2n-1\}$, we have:

$$V_k = (1-\delta)C + \delta V_{k-1}/2 + \delta V_{k+1}/2,$$

with the per-period efficient payoff equal to

$$C = pv(H,H) + qv(H,L) + (1/2 - p - q)v(L,L).$$

For $k = 0$ and $2n$, we have:

$$V_0 = \delta V_1,$$

and

$$V_{2n} = (1-\delta)D + \delta V_{2n-1},$$

where

$$D = 2pv(H,H) + qv(H,L) + (1-2p-2q)v(L,L) + qv(L,H).$$

2.2.3 Payoff Efficiency and Incentive Constraints

We can now demonstrate the efficiency and incentive compatibility of the prescribed strategies:

Proposition 3 *For any given n and $\lambda > 0$, there is a $\overline{\delta} > 0$ such that for every $\delta < \overline{\delta}$, we have that V_k is no lower than a $(2n-1)/4n$ share of the efficient total payoff $2C$ by more than λ for all $k = 0, ..., 2n$.*

Proof. As in the previous application, the proposition follows from the Ergodic Theorem. Indeed, the ergodic probabilities π_k, $k = 0, ..., 2n$, must satisfy the following system of linear equations:

$$\begin{bmatrix} 1 & -1/2 & 0 & . & . & . & 0 \\ -1 & 1 & -1/2 & . & & & . \\ 0 & . & . & . & . & & . \\ . & . & . & . & . & . & . \\ . & & . & . & . & . & 0 \\ . & & & . & -1/2 & 1 & -1 \\ 0 & . & . & . & 0 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} \pi_{2n} \\ . \\ \\ . \\ \\ . \\ \pi_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ . \\ . \\ . \\ \\ 0 \end{bmatrix}.$$

This yields $\pi_0 = \pi_{2n} = 1/4n$ and $\pi_1 = ... = \pi_{2n-1} = 1/2n$. So, players play inefficient actions only in a probability close to $1/2n$. ■

Proposition 4 *For every n, there is a $\overline{\delta} < 1$ such that for every $\delta > \overline{\delta}$, the prescribed chip-strategy profile is an equilibrium of the repeated game.*

Proof. We will begin by showing that players have incentives to report their types truthfully. Consider first what they gain or lose in the current period by misreporting. Player 1 loses

$$(1-\delta)\left\{\frac{p}{2p+q}v(H,H)+\frac{q/2}{2p+q}v(H,L)\right\}$$

by reporting L instead of H, and gains

$$(1-\delta)\left\{\frac{q/2}{1-2p-q}v(L,H)+\frac{1/2-p-q}{1-2p-q}v(L,L)\right\}$$

by reporting H instead of L. Indeed, consider the first formula. When the player's signal is H, she assigns probability $2p/(2p+q)$ to her opponent having signal H; in this case the player loses $v(H,H)$ with probability $1/2$ by reporting L versus H. Similarly, she assigns probability $q/(2p+q)$ to her opponent having signal L; in this case the player loses $v(H,L)$ with probability $1/2$ by reporting L versus H.

In turn, player 1 increases and decreases, respectively, her chance of having $k+1$ chips instead of having $k-1$ chips by

$$\frac{p+q/2}{2p+q}=\frac{1}{2}\text{ and }\frac{1/2-p-q/2}{1-2p-q}=\frac{1}{2}.$$

Thus, we need to show that

$$(1-\delta)\left\{\frac{p}{2p+q}v(H,H)+\frac{q/2}{2p+q}v(H,L)\right\}>\delta\frac{1}{2}(V_{k+1}-V_{k-1}), \tag{4}$$

and

$$(1-\delta)\left\{\frac{q/2}{1-2p-q}v(L,H)+\frac{1/2-p-q}{1-2p-q}v(L,L)\right\}<\delta\frac{1}{2}(V_{k+1}-V_{k-1}), \tag{5}$$

for $k=1,...,2n-1$.

Let $\Delta_k=V_k-V_{k-1}$ for $k=1,...,2n$. Then for $k=2,...,2n-1$, we have

$$\Delta_k=\delta\Delta_{k-1}/2+\delta\Delta_{k+1}/2.$$

At $\delta=1$, we have $\Delta_k=0$ and

$$\frac{\partial\Delta_k}{\partial\delta}=\frac{1}{2}\frac{\partial\Delta_{k-1}}{\partial\delta}+\frac{1}{2}\frac{\partial\Delta_{k+1}}{\partial\delta}.$$

For $k=1$ and $2n$, we have

$$\Delta_{2n}=(1-\delta)[D-C]+\delta\Delta_{2n-1}/2-\delta\Delta_{2n}/2$$

and

$$\frac{\partial \Delta_{2n}}{\partial \delta} = -\frac{2[D-C]}{3} + \frac{1}{3}\frac{\partial \Delta_{2n-1}}{\partial \delta};$$
$$\Delta_1 = (1-\delta)C + \delta\Delta_2/2 - \delta\Delta_1/2$$

and

$$\frac{\partial \Delta_1}{\partial \delta} = -\frac{2C}{3} + \frac{1}{3}\frac{\partial \Delta_2}{\partial \delta}.$$

This implies that $\partial\Delta_k/\partial\delta$ is a weighted average of C and $D-C$ for $k = 1, ..., 2n$. By the Implicit Function Theorem, this is also true, in approximation, for the expressions on the RHS of (4) and (5) multiplied by 2. Recalling the values of C, D, by (4) and (5), it suffices to show that

$$\frac{q}{1-2p-q}v(L,H) + \frac{1-2p-2q}{1-2p-q}v(L,L) <$$
$$< 2pv(H,H)+2qv(H,L)+(1-2p-2q)v(L,L),\ 2pv(H,H)+2qv(L,H)+(1-2p-2q)v(L,L) <$$
$$< \frac{2p}{2p+q}v(H,H) + \frac{q}{2p+q}v(H,L).$$

To show this, notice that each of the four expressions is a weighted average of $v(H,H)$, $v(H,L)$, $v(L,H)$, and $v(L,L)$, where $v(H,H) > v(H,L) > v(L,H) > v(L,L)$. Notice now that the weights of the first expression are first-order stochastically dominated by both the weights of the second expression (the first in the second display) and the weights of the third expression (the second in the second display). In turn, both the weights of the second expression and the weights of the third expression are first-order stochastically dominated by the weights of the fourth expression.

Thus, players have incentives to report their types truthfully. Players have no incentive to deviate at the bidding stage either, because if such a deviation changed the auction outcome, it would be detected; a player who was supposed to be the loser would be the winner (or the other way around).[8] This would mean switching to a stage-game Bayesian Nash equilibrium, in which the payoff of each player is strictly lower than one half of the efficient payoff. This is a worse outcome compared to playing chip strategies where the continuation payoff V_k converges to one half of the efficient payoff for all $k = 0, ..., 2n$ given that the discount factor converges to 1 and n is taken sufficiently large. ■

[8] Here we disregard as profitable any deviations to bids that are more than 0 but less than ρ.

3 Result

We will now introduce the general setting. Let G be a two-player game. Denote by A_i the set of actions, and by T_i the set of types of player $i = 1, 2$; denote by $u_i : A_1 \times A_2 \times T_1 \times T_2 \to \mathbb{R}$ the payoff function of player i. In this section, we consider only *symmetric* games, i.e., we assume that $A_1 = A_2$, $T_1 = T_2$, and $u_1(a_1, a_2, t_1, t_2) = u_2(a_2, a_1, t_2, t_1)$. Players are expected-payoff maximizers and discount future payoffs by a common discount factor $\delta < 1$. The payoffs are normalized by the factor of $1 - \delta$. We allow players to communicate, i.e., to send cheap-talk messages at the beginning of each period.[9]

We analyze only games in which each player's type set has exactly two elements, i.e., $|T_i| = 2$ for $i = 1, 2$. We denote these two types by H and L. We assume that types evolve according to a Markov process, i.e., the type of a player in the following period is equal to the current type with probability $p \in [1/2, 1)$, and is equal to the other type with the remaining probability. In period 0, the probability distribution over the types is fifty-fifty. Types are independent across players.

The efficient action profile (i.e., the action profile maximizing the sum of payoffs) is specified to be symmetric for symmetric type profiles, i.e., if (a_{LH}, a_{HL}) is the specified efficient action profile for type profile (L, H), then (a_{HL}, a_{LH}) is the specified action profile for type profile (H, L). Finally, we assume that for type profiles (H, H) and (L, L), the same actions are specified for both players in the efficient action profile. These actions are denoted by a_{HH} and a_{LL}, respectively.[10] We denote by v the (ex ante expected) total

[9] In some games, including the applications to favor exchange or repeated oligopoly, communication is not necessary. However, in the general case players would not be able to coordinate on efficient actions without learning about the opponents' types.

[10] This assumption is without any loss of generality. If the efficient action profile is asymmetric - take (a_1, a_2) as an example - then (a_2, a_1) is also efficient by symmetry, and it can be replaced with a fifty-fifty lottery over over (a_1, a_2) and (a_2, a_1). Such a lottery does not even require the access to any public randomization device. Instead, players can generate the required public randomization device in communication by randomizing simultaneously over two extraneous messages, with the interpretation that (a_1, a_2) is going to be played if the messages "coincide," and (a_2, a_1) is going to be played if the messages are different.

efficient payoff, that is,

$$v = \frac{u_1(a_{LL}, a_{LL}, L, L) + u_1(a_{LH}, a_{HL}, L, H) + u_1(a_{HL}, a_{LH}, H, L) + u_1(a_{HH}, a_{HH}, H, L)}{2}.$$

The players would each achieve the payoff of $v/2$ if they both would report their types truthfully and then take efficient actions. We will assume, however, that each player obtains a higher current payoff, if she reports one of the types - let us say L - no matter what the player's actual type is. More precisely, we assume that for any player 1's current type t_1 and player 2's type t_2^{-1} in the previous period,

$$E[u_1(a_{Lt_2}, a_{t_2L}, t_1, t_2) \mid t_2^{-1}] - E[u_1(a_{Ht_2}, a_{t_2H}, t_1, t_2) \mid t_2^{-1}] > 0 \tag{6}$$

The inequality guarantees that player 1 prefers reporting L to reporting H. This assumption makes the setting appropriate for using chip strategies. Indeed, each player would always prefer to play as if she was of type L. However, this is not what the other player wants. Therefore, every time the player reports her more-preferred type L, but the opponent reports the other type, the opponent provides the player a favor. In addition, we assume that for $t_2^{-1} = L, H$ the value of the difference in (6) for $t_1 = L$ exceeds that for $t_1 = H$, which makes favors desired from the efficiency perspective.

Finally, to specify the play when one of the players runs out of chips, we assume that there exist action profiles (b_1^L, b_2^L) and (b_1^H, b_2^H), such that player 1 prefers action profile $(b_1^{t_1}, b_2^{t_1})$ to be played when her type is t_1, and which "reward" player 1 at the expense of player 2. To be more precise, we will now provide definitions for the four quantities B, A, B', and A', which are equal to the payoff differences between taking the efficient and "reward" actions. These quantities will determine players' incentives to report truthfully. We begin with quantity B, defined as follows:

$$\begin{aligned}
&B = p^2[u_1(b_1^H, b_2^H, H, L) - u_1(a_{HL}, a_{LH}, H, L)] \\
&+p(1-p)[u_1(b_1^H, b_2^H, H, H) - u_1(a_{HH}, a_{HH}, H, H)] \\
&+(1-p)p[u_1(b_1^L, b_2^L, L, L) - u_1(a_{LL}, a_{LL}, L, L)] \\
&+(1-p)^2[u_1(b_1^L, b_2^L, L, H) - u_1(a_{LH}, a_{HL}, L, H)];
\end{aligned}$$

this quantity is the difference in player 1's expected flow payoff between playing these "reward" action profiles and playing the efficient action profiles, contingent on the type profile being (H, L) in the previous period. We define the second quantity A as follows:

$$
\begin{aligned}
&A = p^2[u_1(a_{LH}, a_{HL}, L, H) - u_1(b_2^H, b_1^H, L, H)] \\
&+p(1-p)[u_1(a_{HH}, a_{HH}, H, H) - u_1(b_2^H, b_1^H, H, H)] \\
&+(1-p)p[u_1(a_{LL}, a_{LL}, L, L) - u_1(b_2^L, b_1^L, L, L)] \\
&+(1-p)^2[u_1(a_{HL}, a_{LH}, H, L) - u_1(b_2^L, b_1^L, H, L)];
\end{aligned}
$$

this quantity is the difference in player 1's expected flow payoff between playing the efficient action profiles and playing (b_2^L, b_1^L) or (b_2^H, b_1^H), as chosen by player 2, contingent on the type profile being (L, H) in the previous period. Note that $A > B$, since the total payoff when playing "reward" actions is lower that the total efficient payoff. Finally, the quantities B', and A' are defined as:

$$
\begin{aligned}
&B' = p(1-p)[u_1(b_1^L, b_2^L, L, H) - u_1(a_{LH}, a_{HL}, L, H)] \\
&+p^2[u_1(b_1^L, b_2^L, L, L) - u_1(a_{LL}, a_{LL}, L, L)] \\
&+(1-p)^2[u_1(b_1^H, b_2^H, H, H) - u_1(a_{HH}, a_{HH}, H, H)] \\
&+p(1-p)[u_1(b_1^H, b_2^H, H, L) - u_1(a_{HL}, a_{LH}, H, L)]
\end{aligned}
$$

- this is the difference in player 1's expected payoff between playing the efficient action profiles and playing (b_1^L, b_2^L) or (b_1^H, b_2^H), contingent on the type profile being (L, L) in the previous period - and

$$
\begin{aligned}
&A' = p(1-p)[u_1(a_{LH}, a_{HL}, L, H) - u_1(b_2^H, b_1^H, L, H)] \\
&+(1-p)^2[u_1(a_{LL}, a_{LL}, L, L) - u_1(b_2^L, b_1^L, L, L)] \\
&+p^2[u_1(a_{HH}, a_{HH}, H, H) - u_1(b_2^H, b_1^H, H, H)] \\
&+p(1-p)[u_1(a_{HL}, a_{LH}, H, L) - u_1(b_2^L, b_1^L, H, L)]
\end{aligned}
$$

- this is the difference in player 1's expected payoff between playing the efficient action profiles and playing (b_2^L, b_1^L) or (b_2^H, b_1^H), as chosen by player 2, contingent on the type profile being (H, H) in the previous period.

We now make two assumptions:

Assumption I: Expressions (6) for $t_1 = L$ and $t_2^{-1} = L, H$ are greater than $A/2p$, and expressions (6) for $t_1 = H$ and $t_2^{-1} = L, H$ are smaller than $B/2p$.

Assumption II: Expression (6) for $t_1 = L$ and $t_2^{-1} = L$ is greater than B', and expression (6) for $t_1 = H$ and $t_2^{-1} = H$ is smaller than A'.

Assumptions I and II require some explanation. Although they look complicated, they simply mean that a player is willing to receive a favor and give away a chip only when it is efficient to do so. It is easier to first understand the assumptions, and the necessity of making them, in the case when types evolve over time in the i.i.d. manner, in which case the factor of $2p = 1$.

Chip strategies must discipline players to report their types truthfully. These strategies do so by letting player 1 play (b_1^L, b_2^L) and (b_1^H, b_2^H), when player 2 runs out of chips, and letting player 2 play the symmetric action profiles when player 1 runs out of chips. In these limit cases, players gain B (or B') or lose A (or A'), compared to the case in which the number of chips is strictly between the limits. Thus, these gain and loss must be large enough to deter players from reporting type L when the actual type is H, but cannot be too large, because that would give them incentives for reporting type H when the actual type is L. Assumptions I and II ensure the right incentives in the i.i.d. case.

In the Markov case, a factor of $2p$ appears in Assumption I because the play returns to a limit state with either 0 chips or 2n chips with some persistence, after the last chip has been exchanged. This is compared to a one-period gain or loss in terms of flow payoffs. This factor does not appear in Assumption II, because it only applies to: (i) the state in which player 1 has $2n$ chips and the type profile in the previous period was (L, L), and (ii) the state in which player 1 has no chip and the type profile in the previous period was (H, H). In turn Assumption I applies to the state with $2n$ chips when the type profile in the previous period was (H, L), and to the state with no chip when the type profile in the previous period was (L, H). On the equilibrium path, the number of chips changes when type profiles are (H, L) and (L, H), and does not change with type profiles (L, L) and (H, H). There is thus some persistence in the former case, but there is no persistence in the latter case. In addition, Assumption II incentivizes player 1 for reporting L truthfully only in a state with $2n - 1$ chips, and it incentivizes player 1 for reporting H truthfully in a state with 1 chip. This is the reason that Assumption II has only one-way inequalities.

Finally, we assume that

Assumption III: There exists a "bad" dynamic-game equilibrium, that is, an equilibrium in which the payoff of a player is strictly lower than $v/2$.

The *simple chip strategies* are defined as follows:

- There are $2n$ chips, which are distributed between the two players.
- If player 1 has k chips, and $k \neq 0$ or $2n$, then the players take the efficient action profile: (a_{LH}, a_{HL}), (a_{HL}, a_{LH}), (a_{HH}, a_{HH}) or (a_{LL}, a_{LL}). The reported type profile determines which of the four action profiles will be taken.
- If either (a_{HH}, a_{HH}) or (a_{LL}, a_{LL}) is played, then the distribution of chips remains unaltered. If either (a_{LH}, a_{HL}) or (a_{HL}, a_{LH}) is played, then player 1 or player 2, respectively, gives a chip to the opponent.
- In the limit state $k = 2n$, the action profile (b_1^L, b_2^L) or (b_1^H, b_2^H) is played, depending on the report of player 1, and player 1 gives player 2 a chip. In the limit state $k = 0$, the action profile (b_2^L, b_1^L) or (b_2^H, b_1^H) is played, depending on the report of player 2, and player 2 gives player 1 a chip.
- Finally, the bad equilibrium is played off the equilibrium path.

We obtain the following result:

Theorem 3 *For every $\lambda > 0$, there is a $\underline{\delta} < 1$ such that for every $\delta > \underline{\delta}$, there is an equilibrium in simple chip strategies in which players' discount factor is δ, such that the ex ante payoff of each player in this equilibrium exceeds $v/2 - \lambda$.*

Proof. The proof can be found in Appendix A. ■

4 Applications, part II

Our theorem has numerous applications. We will now discuss several of them.

4.1 Spulber's duopoly

Consider a repeated version of Spulber's (1995) duopoly model, in which two firms meet in periods $t = 1, 2...$ Each firm's cost of producing one unit of a good takes the value $c = \underline{c}$ or $\overline{c}$, and follows a first-order Markov process. If the cost in some period is $\underline{c}$, then in the following period it will be $\underline{c}$ with probability $p \in [1/2, 1)$, and $\overline{c}$ with the remaining probability. Similarly, if the cost in some period is $\overline{c}$, then in the following period it will be $\overline{c}$ with probability p, and $\underline{c}$ with the remaining probability. In any single period, the costs of the two firms are independent random variables.

In every period of this dynamic game, firms select prices simultaneously. A single consumer is willing to pay up to $r > \overline{c} > \underline{c}$ dollars for one unit of the good, and buys from the firm that offers a lower price; if the two prices are equal, she buys from each firm with a fifty-fifty chance.

Firms are expected-profit maximizers and discount future payoffs by a common discount factor $\delta < 1$. In period 0, the cost of each firm takes the value $\underline{c}$ or $\overline{c}$ with a fifty-fifty chance. Then, the efficient, or most collusive, total payoff of the two firms is

$$v = r - \frac{3}{4}\underline{c} - \frac{1}{4}\overline{c}.$$

Our main result implies that:

Theorem 4 *For every $\lambda > 0$, there is a $\underline{\delta} < 1$ such that for every $\delta > \underline{\delta}$, there is a simple chip-strategy equilibrium of the dynamic game in which the firms' discount factor is δ such that the ex ante payoff of each firm in this equilibrium exceeds $v/2 - \lambda$.*

In short, the idea behind our chip strategies can be described as follows. On the equilibrium path, if in some period, one firm charges a lower price than the other, then it serves the consumer, but gives an implicit chip to the other firm. And if one of the firms has no more chips, it lets the other firm serve the consumer for one period, and receives one chip for this favor. Off the equilibrium path, firms play a bad dynamic-game equilibrium. In terms of our general result, this means that the firms charge prices $a_{LH} = a_{LL} = r - \rho$, $a_{HL} = a_{HH} = r$, where ρ is an infinitesimal number, and $(b_1^L, b_2^L) = (b_1^H, b_2^H) = (r - \rho, r)$. Assumptions I and II reduce to just saying that $(r - \underline{c})/2 > (r - \overline{c})/2 > 0$. Indeed, given the last period's type profile being (H, L), the difference in player 1's expected payoff between playing (b_1^L, b_2^L) and (b_1^H, b_2^H) and playing the efficient action profiles is

$$B = p^2[r - \bar{c}] + (1-p)p\left[\frac{r - \underline{c}}{2}\right] + p(1-p)\left[\frac{r - \bar{c}}{2}\right] \in \left(2p\frac{r - \bar{c}}{2}, 2p\frac{r - \underline{c}}{2}\right),$$

where ρ is taken to be 0. The difference in player 2's expected payoff between playing the efficient action profiles and playing (b_1^L, b_2^L) and (b_1^H, b_2^H) is

$$A = p^2[r - \underline{c}] + p(1-p)\left[\frac{r - \underline{c}}{2}\right] + p(1-p)\left[\frac{r - \bar{c}}{2}\right] \in \left(2p\frac{r - \bar{c}}{2}, 2p\frac{r - \underline{c}}{2}\right).$$

Similarly,

$$\begin{aligned} B' &= p^2\left[\frac{r - \underline{c}}{2}\right] + (1-p)^2\left[\frac{r - \bar{c}}{2}\right] + p(1-p)[r - \bar{c}] \\ &= p^2\left[\frac{r - \underline{c}}{2}\right] + (1-p^2)\left[\frac{r - \bar{c}}{2}\right] \in \left(\frac{r - \bar{c}}{2}, \frac{r - \underline{c}}{2}\right) \end{aligned}$$

and

$$\begin{aligned} A' &= p(1-p)[r - \underline{c}] + (1-p)^2\left[\frac{r - \underline{c}}{2}\right] + p^2\left[\frac{r - \bar{c}}{2}\right] \\ &= (1-p^2)\left[\frac{r - \underline{c}}{2}\right] + p^2\left[\frac{r - \bar{c}}{2}\right] \in \left(\frac{r - \bar{c}}{2}, \frac{r - \underline{c}}{2}\right). \end{aligned}$$

Off equilibrium, i.e., when a firm charges a price other than r or $r - \rho$, or does not charge the prescribed price in states $k = 0$ or $2n$, the firms switch to playing a "bad" equilibrium, in which both firms obtain relatively low payoffs. The bad equilibrium used in this particular game can be, for example, the worst carrot-and-stick equilibrium from Athey and Bagwell (2008).

In Athey and Bagwell's carrot-and-stick equilibria, there are two states. In the war state, all firms choose a price γ lower than r; and in the reward state, all firms charge price r. Firms begin in the war state. In the war state, if both firms choose price $\gamma < r$, the firms switch to the reward state with a probability μ, and return to the war state with the remaining probability. In the reward state, if both firms choose price r, the firms remain in the reward state with probability 1. In each period, if any firm charges a price other than the prescribed price, the firms switch to the war state with probability 1.

The off-equilibrium payoff of each firm, when the discount factor converges to 1, is bounded by $r/2 - \underline{c}/2 - \bar{c}/2$, regardless of the current cost profile, which is less than the efficient payoff.[11] Notice that in this application, chip strategies require no explicit communication, because types are revealed through actions.

4.2 Taking turns

Suppose two individuals have to perform an unpleasant duty, such as cleaning their shared apartment. This task must be performed in every single period. The cost of performing it is $c_i = \underline{c}$ or $\overline{c}$, $i = 1, 2$, where $\underline{c} < \overline{c} \in (1, 2)$. The single player's payoff from having the task performed is 0, and the payoff is equal to -1 otherwise. The costs are independent across individuals and Markov over time. It is efficient if the task is always performed by an individual with lower cost.

This model has been studied by Leo (2015) who assumed that the costs are i.i.d. over time, but allowed them to have more than two values - for example, to be continuously distributed on $(1, 2)$. He showed numerically that the total payoff achieved by the simple chip mechanism (with a sufficiently large number of chips, and for the parameter values assumed in the numeric exercise) converges to an appropriately defined second-best outcome.

The approximate efficiency and incentive compatibility of the simple chip strategies in Leo's (2015) model with two possible cost values follow from our result. The efficient actions are: a_{LH} is to perform or volunteer, a_{HL} is to do nothing, a_{HL} and a_{LH} are the symmetric actions, and $(a_{LL}, a_{LL}) = (a_{HH}, a_{HH})$ is the action profile in which each individual performs the duty with probability $1/2$.[12] Finally, the action profile $(b_1^L, b_2^L) = (b_1^H, b_2^H)$ requires individual 2 to perform the task no matter what the cost profile, while the action profile $(b_2^L, b_1^L) = (b_2^H, b_1^H)$ requires individual 1 to perform the task no matter what the cost profile. The bad equilibrium is not cleaning the apartment.

In a similar model, players have in each period some good, let us say an apple, to be consumed by one of them. The private type (high or low) of a player $i = 1, 2$ is the value of consuming the apple. That is, if player i consumes the apple, her utility in that period is equal to her type. In each period each player announces her type, and the player with

[11] The bound follows from the fact that strategies are independent of the firms' costs.

[12] This action profile is feasible with a public randomization device, but does not require access to any public randomization device, since players can generate the fifty-fifty lottery in communication.

the higher type consumes the apple; if both players announce the same type, the apple is allocated by the fifty-fifty lottery. Players also have the option of not allowing anybody to consume the apple. Again, simple chip equilibria approximate the efficient outcome.

4.3 Other applications

4.3.1 Repeated auctions

In our result, we assumed that types are independent across players, and evolve according to a Markov process. Alternatively, our result could be extended to the setting with interdependent types, with type profile being symmetric across players and i.i.d. over time. The proof parallels very closely the proof for the repeated auctions.

We conjecture, but have not shown formally, that our theorem generalizes to symmetric probability distribution over types, which are not necessarily independent across players, and evolve according to a Markov process.

4.3.2 Other versions of favor exchange

In Section 6, we will continue the discussion of the favor exchange model from Section 2. We consider here a distinct version of the model. In every period, both players obtain a dollar, which can be consumed or transferred. The transfer creates $\gamma > 1$ value for the opponent when her type is L, but only the value of $\mu < 1$ when the opponent's type is H. The distribution over type profiles is the same as in Section 2 with L corresponding to the opponent receiving \$1. If both players report type L, then players know that one of them is lying. However, this does not matter for incentive compatibility. The player who has no chips is prescribed to transfer her income to the opponent even if this transfer is inefficient, while the opponent is prescribed to consume her income herself. Then $B = (1-p) + p\mu$ and $A = (1-p)\gamma + p$, and since these values fall between μ and γ,[13] our result from Section 3 implies that the simple chip strategies approximate the efficient outcome.

[13] By reporting H, type L loses γ because she is not provided a favor, and by reporting L, type H gains μ because she is provided a favor although she should not be.

4.3.3 Dynamic cheap talk

Suppose a sender recommends a receiver one of two actions, say, to invest either in a risky or in a safe asset. The sender is equally likely to obtain the signal that the receiver should invest in the risky assets as well as the signal that the receiver should invest in the safe asset. The sender always prefers that the receiver invests in the risky asset, though she gets a higher payoff conditionally on the signal that the receiver should invest in the risky asset. The efficiency requires the "right" action of the receiver, and the receiver will take the safe action in the absence of any advice from the sender.

Since the game is asymmetric, our results do not apply directly. However, the approximate efficiency and incentive compatibility follow from analogous arguments. In such a simple chip equilibrium, the sender recommends the right investment, and the receiver follows the sender's advice. Finally, the receiver always takes the risky action if the sender collects all the chips, and always takes the safe action if the receiver collects all the chips.

5 Asymmetric games

We have been assuming the symmetry of a stage game. This assumption is essential if we restrict attention to simple chip strategies. In this section, we will use an example to show that the efficient outcome may not be approximated in simple chip strategies in asymmetric games. Then, we will introduce a slightly more general class of *random chip strategies*, and show that the efficient outcome in this example can be approximated in this more general class. [14] Random chip strategies differ from simple chip strategies only in that they allow for chips to be transferred with some probability.

Consider the game in which the signal of player 1 determines which actions are efficient. This is a feature of numerous settings, including some asymmetric versions of favor-exchange models and repeated auctions. More specifically, suppose that in each period players have an indivisible good, such as an apple, to share. If player 1's signal is L, then player 2 values the apple more, but if player 1's signal is H, player 1 values the apple more. (For example, the value of the apple is constant for player 2, but varies over time

[14] The general result for asymmetric games is shown in our companion paper, Olszewski and Safronov (2015).

for player 1.) Player 1 obtains signal H with probability θ; signal L is obtained with the remaining probability. Suppose further that player 1 can suggest who gets to consume the apple, and player 2 can agree or disagree. If player 2 disagrees, no player is allowed to consume the apple. Otherwise, the apple is allocated according to player 1's suggestion.

If $\theta = 1/2$, then one can easily show that the efficient outcome can be approximated by simple chip strategies in which player 1 obtains the apple and gives player 2 a chip when she says "mine." Player 2 obtains the apple and gives player 1 a chip when player 1 says "yours". This is contingent on the number of chips not reaching a limit value. For the limit value, player 1 says "mine" or "yours," depending on who has all the chips, but independently of her signal. If player 2 has all the chips but player 1 says "mine" (which can happen only off the equilibrium path), this triggers the phase in which no player is ever allowed to consume an apple.

Proposition 5 *Suppose that $\theta \neq 1/2$. Then there exists a bound $K > 0$ such that for all $\delta < 1$ the ex ante expected total payoff of any equilibrium in simple chip strategies is lower than the efficient total payoff by at least K.*

Proof. Let $\theta > 1/2$. Denote by k the number of chips that player 1 has. Then, the transition probability from the state with k chips to the state with $k-1$ chips is greater than $1/2$, while that to the state with $k+1$ chips is smaller than $1/2$. Since the transition probability from the state with $2n$ chips to the state with $2n-1$ chips, and from the state with 0 chips to the state with 1 chip, is equal to 1, the ergodic probability distribution over states π_k, $k = 0, 1, ..., 2n$, must satisfy the following system of linear equations:

$$\begin{bmatrix} 0 & 1-\theta & 0 & . & . & . & 0 \\ 1 & 0 & 1-\theta & . & & & . \\ 0 & \theta & 0 & 1-\theta & . & & . \\ . & . & . & . & . & . & . \\ . & & . & \theta & 0 & 1-\theta & 0 \\ . & & & . & \theta & 0 & 1 \\ 0 & . & . & . & 0 & \theta & 0 \end{bmatrix} \begin{bmatrix} \pi_{2n} \\ . \\ \\ . \\ \\ . \\ \pi_0 \end{bmatrix} = \begin{bmatrix} \pi_{2n} \\ . \\ \\ . \\ \\ . \\ \pi_0 \end{bmatrix}.$$

This system can be solved by the Gauss-Jordan method. This yields

$$\pi_{2n} = (1-\theta)\left(\frac{1-\theta}{\theta}\right)^{2n-2}\left(\frac{1}{\theta}\right)\cdot$$
$$\cdot\left\{1+\left(\frac{1}{\theta}\right)\left[1+\left(\frac{1-\theta}{\theta}\right)+...+\left(\frac{1-\theta}{\theta}\right)^{2n-2}\right]+(1-\theta)\left(\frac{1-\theta}{\theta}\right)^{2n-2}\left(\frac{1}{\theta}\right)\right\}^{-1}.$$

Thus the probability of being in an inefficient state $2n$ is

$$\pi_{2n} \longrightarrow_{n\to\infty} \frac{2\theta-1}{2\theta} > 0.$$

This yields the required bound, since the efficient total payoff could be approximated in simple chip strategies only when the probability of being in an inefficient state limits to 0.

■

We will now show that the efficient outcome can be approximated by *random chip strategies*. These strategies allow for using a public randomization device. Player 1 obtains the apple and gives player 2 a chip when she says "mine," but only with probability q such that $q\theta = 1-\theta$. Player 2 obtains the apple and gives player 1 a chip (with probability 1) when player 1 says "yours". This is contingent on the number of chips not reaching a limit value. For a limit value, the apple is allocated independent of player 1's signal. That is, player 1 obtains the apple with probability r_{2n} if player 2 has no chips, and player 2 obtains the apple with the remaining probability of $1-r_{2n}$. If player 1 has no chips, player 1 obtains the apple with probability r_0, and player 2 obtains the apple with the remaining probability of $1-r_0$. With probability $1-\theta$, one chip is returned to the player who currently has no chips. Off the equilibrium path, no player is ever allowed to consume an apple.

Proposition 6 *For every $\lambda > 0$, there is a $\underline{\delta} < 1$ such that for every $\delta > \underline{\delta}$, there is a random chip-strategy equilibrium of the repeated game in which the players' discount factor is δ, such that the ex ante expected total payoff in this equilibrium cannot be lower than the efficient total payoff by more than λ.*

Proof. The fact that the random chip strategies generate the total payoff that converges to the efficient total payoff (as the number of chips $2n$ diverges to ∞) follows from the Ergodic Theorem by the arguments we have used several times previously.

To show that incentive constraints are satisfied, we first evaluate the continuation payoff of player 1; this payoff will be denoted by V_k. For $k \in \{1, ..., 2n-1\}$, we have:

$$V_k = \theta\{(1-\delta)A_H + \delta(qV_{k-1} + (1-q)V_k\} + (1-\theta)\delta V_{k+1},$$

where A_t is the value of an apple for type $t = H, L$ of player 1. For $k = 0$ and $2n$, we have:

$$V_0 = r_0[(1-\delta)\theta A_H + (1-\delta)(1-\theta)A_L] + \theta\delta V_0 + (1-\theta)\delta V_1,$$

$$V_{2n} = r_{2n}[(1-\delta)\theta A_H + (1-\delta)(1-\theta)A_L] + \theta\delta V_{2n} + (1-\theta)\delta V_{2n-1}.$$

It follows that for the difference $\Delta_k = V_k - V_{k-1}$:

$$\Delta_k = \theta q\delta\Delta_{k-1} + \theta(1-q)\delta\Delta_k + (1-\theta)\delta\Delta_{k+1}$$

for $k = 2, ..., 2n-1$, which yields (since $1-\theta = q\theta$)

$$\frac{\partial\Delta_k}{\partial\delta} = \frac{1}{2}\frac{\partial\Delta_{k-1}}{\partial\delta} + \frac{1}{2}\frac{\partial\Delta_{k+1}}{\partial\delta}, \tag{7}$$

where the derivatives are evaluated at $\delta = 1$; in addition, note that $\Delta_k = 0$ for $\delta = 1$.

Similarly,

$$\Delta_1 = (1-r_0)(1-\delta)\theta A_H - r_0(1-\delta)(1-\theta)A_L + (1-\theta)\delta\Delta_2 + \theta\delta(1-q)\Delta_1,$$

$$\frac{\partial\Delta_1}{\partial\delta} = \frac{-(1-r_0)\theta A_H + r_0(1-\theta)A_L}{2-2\theta} + \frac{1}{2}\frac{\partial\Delta_2}{\partial\delta}$$

and

$$\Delta_{2n} = -(1-r_{2n})(1-\delta)\theta A_H + r_{2n}(1-\delta)(1-\theta)A_L + (1-\theta)\delta\Delta_{2n-1} + \theta\delta(1-q)\Delta_{2n},$$

$$\frac{\partial\Delta_{2n}}{\partial\delta} = \frac{(1-r_{2n})\theta A_H - r_{2n}(1-\theta)A_L}{2-2\theta} + \frac{1}{2}\frac{\partial\Delta_{2n-1}}{\partial\delta}.$$

Take r_0 and r_n such that

$$\frac{-(1-r_0)\theta A_H + r_0(1-\theta)A_L}{2-2\theta} = \frac{-1}{2(1+q)}A_H,$$

$$\frac{(1-r_{2n})\theta A_H - r_{2n}(1-\theta)A_L}{2-2\theta} = \frac{-1}{2(1+q)}A_L.$$

Such $r_0, r_{2n} \in (0,1)$ exist, since the LHS falls below the RHS for $r_0 = 0$, but the LHS exceeds the RHS for $r_0 = 1$; also, the LHS exceeds the RHS for $r_{2n} = 0$, but the LHS falls below the RHS for $r_{2n} = 1$. Thus,

$$\frac{\partial \Delta_1}{\partial \delta} = \frac{-1}{2(1+q)} A_H + \frac{1}{2} \frac{\partial \Delta_2}{\partial \delta}, \tag{8}$$

and

$$\frac{\partial \Delta_{2n}}{\partial \delta} = \frac{-1}{2(1+q)} A_L + \frac{1}{2} \frac{\partial \Delta_{2n-1}}{\partial \delta}. \tag{9}$$

Equations (7)-(9) imply that Δ_k for $k = 1, ..., 2n$, for a sufficiently large discount factor, and in approximation, is a convex combination of $(1-\delta)\frac{1}{1+q}A_L$ and $(1-\delta)\frac{1}{1+q}A_H$. This in turn implies that player 1 has the right incentives. Indeed, by "reporting" H instead of L, player 1 gains A_L, but loses 1 chip with probability $1-q$, and 2 chips with probability q. And by "reporting" L instead of H, player 1 loses A_H, but gains 1 chip with probability $1-q$ and 2 chips with probability q. ■

6 The play in limit states

When a limit number of chips is reached, simple chip strategies prescribe the type-dependent action profiles (b_1^L, b_2^L) and (b_1^H, b_2^H) such that Assumptions I and II are satisfied. This is sufficient for Theorem 3, but necessary only when we want (as the simple chip strategies do) to leave the limit state in one period.

In some stage games, there exist no action profiles (b_1^L, b_2^L) and (b_1^H, b_2^H) with the required property, yet chip strategies achieve efficiency when we allow the play to stay in the limit case for more than one period. The favor exchange model provides an example.[15]

We can replace the action profiles (b_1^L, b_2^L) and (b_1^H, b_2^H) with action profiles that depend on the type of the opponent (player with no chips), that is, with some action profiles (b_1^{LL}, b_2^{LL}), (b_1^{LH}, b_2^{LH}), (b_1^{HL}, b_2^{HL}), and (b_1^{HH}, b_2^{HH}) such that A, B, A', and B' for the new action profiles fall within an appropriate range.

Of course, the opponent must have incentives to truthfully reveal her type. This is the case for some but not all stage games. For some other games, the opponent may

[15] Note first that we can always adjust the game making all actions possible for all types (perhaps, at very large or infinite costs).

have the incentive to always report the same type. In this case, the chip strategies may prescribe staying in the limit case until the other, "unwanted" type is reported.[16] This is exactly the case in the favor-exchange model, in which the action $b_2^{LL} = b_2^{HL}$ (assuming that this is player 2 who already has no chips) is "consume," and the action $b_2^{LH} = b_2^{HH}$ is "transfer"$\dot{}$Player 2 is prescribed to "transfer" if the cost of doing it is a dollar but "consume" if the cost of doing it is infinite. This obviously requires appropriately modified conditions corresponding to A, B, A', and B'.

7 More than two types, more than two players

The case of more than two types turns out to be even more restrictive. We demonstrate below that even random chip strategies may not approximate efficient outcomes.

Consider the model in which two players have an apple in each period. Suppose that there are three possible, equally likely values of the apple - 1, 2, and 3 - that are i.i.d. over time and independent across players.

In the efficient outcome the apple is allocated to the higher type (or randomly if the types are equal). Now, consider an arbitrary chip-strategy profile. Suppose that each player begins the game with n chips. If player i announces type $\widehat{t_i}$ which is higher than type $\widehat{t_j}$ announced by player j, then player i obtains the apple and gives a chip to player j, with a probability that depends on the two announcements. That is, if $\widehat{t_i} > \widehat{t_j}$, then player i gives player j a chip with probability $p_{i,j}$. If the players announce the same type, the state of chips is unchanged, and each player consumes the apple with probability $1/2$.[17] If a player reaches 0 chips, the player's opponent consumes the apple, and gives the player a chip with probability p_r.

We will show that:

Proposition 7 *There exists a value V strictly lower than the efficient payoff, such that any random chip strategies attaining the total payoff higher than V are not incentive compatible, if players are patient enough.*

[16] One can even push this logic further, and prescribe staying in the limit case until the other, "unwanted" type is reported a given number of times.

[17] This assumption is made only for the sake of simplicity.

Proof. The proof of this result can be found in Appendix B. ■

The case of more than two players raises an additional difficulty, namely, groups of players may provide favors to other groups of players, and the contribution of different players from the former groups, as well as the benefits to the players from the latter groups, may be different. This must be reflected in the transition of the distribution of chips in the economy.

Overcoming these difficulties calls for more complicated strategies, which is somewhat in tension with the objective of providing a positive model of playing dynamic games. We have no fully satisfying resolution of this issue. In a companion paper Olszewski and Safronov (2015), we suggest a more complicated version of chip strategies, constructed by imitating the mechanism in d'Aspremont and Gerard-Varet (1979), which approximate the efficient outcomes for a large class of games (with any number of types and players).

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9 Appendix A

It will be convenient to provide first a little more detailed description of the chip strategies on the equilibrium path. In particular, this will allow for introducing our notation. The strategy profile has $4(2n+1)-6$ states. Each state is described by a number $k \in \{0, 1, ..., 2n\}$, and the profile of players' types $t = (t_1, t_2) \in \{L, H\}^2$, as reported in the previous period, with the exception that $k = 0$ implies that $t = (L, H)$ and $k = 2n$ implies that $t = (H, L)$. The number k is interpreted as player 1 having k chips.

In all states except when $k = 0$ or $2n$, if the reported type profile is $t = (t_1, t_2)$, the players are supposed to take the efficient action profile. If $t_1 = t_2$, the value of k does not change. If $t_1 = H$ and $t_2 = L$, then k is replaced with $k+1$ at the end of the current period; and if $t_1 = L$ and $t_2 = H$, then k is replaced with $k-1$.

If the current state has $k = 2n$, then players are supposed to play (b_1^L, b_2^L) or (b_1^H, b_2^H), the choice between the two being determined by the report of player 1, and $k = 2n$ is replaced with $k = 2n-1$ at the end of the current period. If the current state has $k = 0$, then players are supposed to play (b_2^L, b_1^L) or (b_2^H, b_1^H), determined by the report of player 2, and $k = 0$ is replaced with $k = 1$.

Notice that state $k = 0$ can be reached only when type profile $t = (L, H)$ was reported in the previous period, and state $k = 2n$ can be reached only when type profile (H, L) was reported in the previous period.

9.1 Continuation payoffs

Assuming that both players play the prescribed strategies, denote the continuation payoff of player 1 in state k, t by $V_{k,t}$. By the symmetry of our model, the continuation payoff of player 2 can be expressed analogously. These continuation payoffs are computed before the players learn about their types in the current period. For $k \in \{1, ..., 2n-1\}$, we have:

$$\begin{aligned} V_{k,L,H} = {} & (1-\delta)\{p^2 u_1(a_{LH}, a_{HL}, L, H) + p(1-p)u_1(a_{LL}, a_{LL}, L, L) \\ & +(1-p)pu_1(a_{HH}, a_{HH}, H, H) + (1-p)^2 u_1(a_{HL}, a_{LH}, H, L)\} \\ & +\delta\{p^2 V_{k-1,L,H} + p(1-p)V_{k,L,L} + (1-p)pV_{k,H,H} + (1-p)^2 V_{k+1,H,L}\}. \end{aligned} \tag{10}$$

Indeed, the first component of the right-hand side corresponds the payoff contingent on player 1 having type L in the current period, and player 2 having type H; the remaining components correspond to the other type profiles.

Similarly, we obtain:

$$\begin{aligned}V_{k,L,L} = {} & (1-\delta)\{p(1-p)u_1(a_{LH}, a_{HL}, L, H) + p^2 u_1(a_{LL}, a_{LL}, L, L) \\ & +(1-p)^2 u_1(a_{HH}, a_{HH}, H, H) + (1-p)p u_1(a_{HL}, a_{LH}, H, L)\} \\ & +\delta\left\{p(1-p)V_{k-1,L,H} + p^2 V_{k,L,L} + (1-p)^2 V_{k,H,H} + (1-p)pV_{k+1,H,L}\right\};\end{aligned} \tag{11}$$

$$\begin{aligned}V_{k,H,H} = {} & (1-\delta)\{(1-p)p u_1(a_{LH}, a_{HL}, L, H) + (1-p)^2 u_1(a_{LL}, a_{LL}, L, L) \\ & +p^2 u_1(a_{HH}, a_{HH}, H, H) + p(1-p) u_1(a_{HL}, a_{LH}, H, L)\} \\ & +\delta\left\{(1-p)pV_{k-1,L,H} + (1-p)^2 V_{k,L,L} + p^2 V_{k,H,H} + p(1-p)V_{k+1,H,L}\right\};\end{aligned} \tag{12}$$

and

$$\begin{aligned}V_{k,H,L} = {} & (1-\delta)\{(1-p)^2 u_1(a_{LH}, a_{HL}, L, H) + (1-p)p u_1(a_{LL}, a_{LL}, L, L) \\ & +p(1-p) u_1(a_{HH}, a_{HH}, H, H) + p^2 u_1(a_{HL}, a_{LH}, H, L)\} \\ & +\delta\left\{(1-p)^2 V_{k-1,L,H} + (1-p)pV_{k,L,L} + p(1-p) V_{k,H,H} + p^2 V_{k+1,H,L}\right\}.\end{aligned} \tag{13}$$

For $k = 0$ and $2n$, we have:

$$\begin{aligned}V_{0,L,H} = {} & (1-\delta)\{p^2 u_1(b_2^H, b_1^H, L, H) + p(1-p) u_1(b_2^L, b_1^L, L, L) \\ & +(1-p)p u_1(b_2^H, b_1^H, H, H) + (1-p)^2 u_1(b_2^L, b_1^L, H, L)\} \\ & +\delta\left\{p^2 V_{1,L,H} + p(1-p)V_{1,L,L} + (1-p)pV_{1,H,H} + (1-p)^2 V_{1,H,L}\right\},\end{aligned} \tag{14}$$

$$\begin{aligned}V_{2n,H,L} = {} & (1-\delta)\{(1-p)^2 u_1(b_1^L, b_2^L, L, H) + (1-p)p u_1(b_1^L, b_2^L, L, L) \\ & +p(1-p) u_1(b_1^H, b_2^H, H, H) + p^2 u_1(b_1^H, b_2^H, H, L)\} \\ & +\delta\left\{(1-p)^2 V_{2n-1,L,H} + (1-p)pV_{2n-1,L,L} + p(1-p)V_{2n-1,H,H} + p^2 V_{2n-1,H,L}\right\}.\end{aligned} \tag{15}$$

9.1.1 Payoff Efficiency of Prescribed Strategies

We can now demonstrate the efficiency of the prescribed strategies:

Proposition 8 *For every $\lambda > 0$, there are values $\overline{\delta} < 1$ and n_0 such that for every $\delta > \overline{\delta}$ and $n > n_0$, we have that*

$$\left| V_{k,t} - \frac{v}{2} \right| < \lambda$$

for all $k = 0, ..., 2n$ and $t \in \{L, H\}^2$ ($t = (L, H)$ when $k = 0$ and $t = (H, L)$ when $k = 2n$).

Proof. The strategies induce a stochastic Markov chain over states k, t. It is easy to check that in $2n + 2$ periods each state is reached from any other state with positive probability. Thus, by the Ergodic Theorem there exists a probability distribution over states $\{\pi_{k,t} : k, t\}$ such that the probability of being in state k, t after a sufficiently large number of periods is arbitrarily close to $\pi_{k,t}$. This probability distribution is an eigenvector of the transition matrix corresponding to eigenvalue 1.

To estimate the probabilities $\pi_{0,t}$ and $\pi_{2n,t}$, consider first an auxiliary Markov chain with $8n$ states, in which instead of states with $k = 0$ or $k = 2n$ we have four states $\{0, 2n\}$, t for $t \in \{L, H\}^2$. One should think of the "number" $\{0, 2n\}$ as being between the 1 and $2n - 1$. The transitions are as in the Markov chain induced by our strategies, except states $\{0, 2n\}$, t. In state $\{0, 2n\}$, (L, H), the chain transits to: 1, (H, L) with probability $(1 - p)^2$; to $2n - 1$, (L, H) with probability p^2; and to $\{0, 2n\}$, (L, L) and $\{0, 2n\}$, (H, H) with probability $p(1 - p)$ each. In state $\{0, 2n\}$, (H, L), the chain transits to: 1, (H, L) with probability p^2; to $2n - 1$, (L, H) with probability $(1 - p)^2$; and to $\{0, 2n\}$, (L, L) and $\{0, 2n\}$, (H, H) with probability $p(1 - p)$ each. In states $\{0, 2n\}$, (L, L) and $\{0, 2n\}$, (H, H), the chain transits to: 1, (H, L) with probability $p(1 - p)$; to $2n - 1$, (L, H) with probability $p(1-p)$; and to $\{0, 2n\}$, (L, L) and to $\{0, 2n\}$, (H, H) with probability $(1-p)^2$ and p^2, and p^2 and $(1 - p)^2$, respectively.

The ergodic theorem still applies to the new chain. For any $t = (t_1, t_2) \in \{H, L\}^2$, this new chain is completely symmetric across states k, t, where $k = 1, ..., 2n-1$, $\{0, 2n\}$, and therefore for all k the sum of ergodic probabilities $\overline{\pi}_{k,t}$ over $t \in \{L, H\}^2$ is equal to $1/2n$.

Claim 1 *We will show that $\overline{\pi}_{k,t} = 1/8n$ for all k and t.*

Indeed, the current state can be equal to k, (H, H) only when the previous state was equal to k, t. This happens with probability $(1 - p)^2$ if $t = (L, L)$, with probability p^2 if

$t = (H, H)$, and with probability $p(1-p)$ if $t = (H, L)$ or (L, H). This yields

$$\overline{\pi}_{k,(H,H)} = (1-p)p\overline{\pi}_{k,(L,H)} + (1-p)^2\overline{\pi}_{k,(L,L)} + p^2\overline{\pi}_{k,(H,H)} + p(1-p)\overline{\pi}_{k,(H,L)}.$$

Similarly,

$$\overline{\pi}_{k,(L,L)} = p(1-p)\overline{\pi}_{k,(L,H)} + p^2\overline{\pi}_{k,(L,L)} + (1-p)^2\overline{\pi}_{k,(H,H)} + (1-p)p\overline{\pi}_{k,(H,L)}.$$

In turn, the current state can be equal to k, (H, L) only when the previous state was $k-1$, t. (Recall that $k = \{0, 2n\}$ is a number which is one above $k = 2n-1$ and one below $k = 1$.) This happens with probability $p(1-p)$ if $t = (L, L)$ or (H, H), with probability p^2 if $t = (H, L)$, and with probability $(1-p)^2$ if $t = (L, H)$. This yields

$$\overline{\pi}_{k,(H,L)} = (1-p)^2\overline{\pi}_{k-1,(L,H)} + (1-p)p\overline{\pi}_{k-1,(L,L)} + p(1-p)\overline{\pi}_{k-1,(H,H)} + p^2\overline{\pi}_{k-1,(H,L)}.$$

Similarly,

$$\overline{\pi}_{k,(L,H)} = p^2\overline{\pi}_{k+1,(L,H)} + p(1-p)\overline{\pi}_{k+1,(L,L)} + (1-p)p\overline{\pi}_{k+1,(H,H)} + (1-p)^2\overline{\pi}_{k+1,(H,L)}.$$

Due to symmetry across k, any solution of the system of these four equations, which we will call ergodic equations for states k, t, must have all $\overline{\pi}_{k,t}$ equal, which yields $\overline{\pi}_{k,t} = 1/8n$ for all k and t.

We will now return to the analysis of original chain induced by our strategies.

Claim 2 *We will show that the ratio $\pi_{k,t}/\pi_{1,(L,H)}$, for $k = 0, 1$ and all t, is independent of n, and so is the ratio $\pi_{k,t}/\pi_{2n-1,(H,L)}$, for $k = 2n-1, 2n$ and all t.*

Similarly to the proof of the previous claim, we obtain that the following "ergodic" equations:

$$\pi_{0,(L,H)} = p^2\pi_{1,(L,H)} + p(1-p)\pi_{1,(L.L)} + (1-p)p\pi_{1,(H,H)} + (1-p)^2\pi_{1,(H,L)},$$

$$\pi_{1,(H,H)} = (1-p)p\pi_{1,(L,H)} + (1-p)^2\pi_{1,(L,L)} + p^2\pi_{1,(H,H)} + p(1-p)\pi_{1,(H,L)} + (1-p)p\pi_{0,(L,H)},$$

$$\pi_{1,(L,L)} = p(1-p)\pi_{1,(L,H)} + p^2\pi_{1,(L,L)} + (1-p)^2\pi_{1,(H,H)} + (1-p)p\pi_{1,(H,L)} + p(1-p)\pi_{0,(L,H)},$$

and

$$\pi_{1,(H,L)} = (1-p)^2\pi_{0,(L,H)}.$$

Divide each equation by $\pi_{1,(L,H)}$. This yields a system of equations with variables $\pi_{k,t}/\pi_{1,(L,H)}$ for $k=0,1$ and $t \neq (L,H)$. This system has a unique solution, independent of n, and this yields the first part of the claim. Obviously, the second part can be proved in an analogous way.

Claim 3 *We will show that $\pi_{k,t} = \pi_{2,(H,L)} = \pi_{2n-2,(L,H)}$ for $k = 2, ..., 0, ..., 2n-2$ and all t.*

Indeed, notice first that by symmetry $\pi_{2,(H,L)} = \pi_{2n-2,(L,H)}$. Next, observe that except states $2,(H,L)$ and $2n-2,(L,H)$, the ergodic equations for states k, t, where $k = 2, ..., 0, ..., 2n-2$, include no probability $\pi_{0,d}$, $\pi_{1,d}$, $\pi_{2n-1,d}$, or $\pi_{n,d}$, for any $d \in \{L,H\}^2$. This means that for any given $\pi_{2,(H,L)}$ and $\pi_{2n-2,(L,H)}$, the remaining probabilities $\pi_{k,t}$, where $k = 2, ..., 0, ..., 2n-2$, are determined by these ergodic equations.

The original chain induced by our strategies can be obtained from the chain used in Claim 1 by renaming state $\{0,2n\}$, (L,H) as 0, (L,H), renaming state $\{0,2n\}$, (H,L) as $2n$, (H,L), changing appropriately the transition probabilities in the two states, and removing states $\{0,2n\}$, (L,L) and $\{0,2n\}$, (H,H). This means that for any given $\overline{\pi}_{2,(H,L)} = \overline{\pi}_{2n-2,(L,H)}$, the remaining probabilities $\overline{\pi}_{k,t}$, where $k = 2, ..., 0, ..., 2n-2$, are determined by the same ergodic equations as in the case of the original chain induced by our strategies. And since by Claim 1 $\overline{\pi}_{k,t} = \overline{\pi}_{2,(H,L)} = \overline{\pi}_{2n-2,(L,H)}$ for $k = 2, ..., 0, ..., 2n-2$ and all t, the same must be true when $\overline{\pi}$'s are replaced with π's.

Finally, the ergodic equations for $2,(H,L)$ and $2n-2,(L,H)$ imply that $\pi_{2,(H,L)}$ is a weighted average of $\pi_{1,t}$ for $t \in \{L,H\}^2$ and $\pi_{2n-2,(L,H)}$ is a weighted average of $\pi_{2n-1,t}$ for $t \in \{L,H\}^2$ with weights p^2, $(1-p)^2$, $p(1-p)$, $p(1-p)$. This together with Claim 2 and 3 imply that any two probabilities $\pi_{k,t}$, where $k = 0, 1, 2n-1, 2n$ and $t \in \{L,H\}^2$ are proportional , and the coefficients of proportionality depend only on p, that is, are independent of n. Thus, all probabilities converge to zero as n diverges to infinity.

Since our strategies are inefficient only in states $0,(L,H)$ and $2n,(H,L)$, this means that as $1-\delta$ is sufficiently close to 0, the inefficiency is approximately proportional to the sum of the ergodic probabilities of states $0,(L,H)$ and $2n,(H,L)$. Therefore it disappears when n diverges to infinity. By symmetry of the strategies the payoff of each player is close to a half of the efficient payoff. ■

9.1.2 Incentive Constraints

We will now turn to verifying the incentive constraints. The constraints for the states in which $k = 0$ or $2n$ are immediate, since no player wants to trigger the bad equilibrium, the action profiles b^L and b^H are independent of the report of the player with no chips, and the player with all chips can choose between b^L and b^H. Thus, consider the states k such that $0 < k < 2n$. For each of these $8n - 4$ states, there are two constraints to check: one for the player with type L and one for the player with type H.

Suppose that the play is in state k, t. We will check the constraints for player 1. (By symmetry, this will imply the constraints for player 2.) Consider first the effect of playing the prescribed strategies, compared to a deviation, on the state in the following period. If $t_1 = L$, by reporting honestly player 1 will be in a state with one fewer chip compared to reporting state H, but the distribution over the next period's state $t^{+1} \in \{L, H\}^2$ will be exactly the same under the two possible reports. Similarly, if $t_1 = H$, player 1 will be in a state with one more chip by reporting honestly compared to reporting L, but the distribution over $t^{+1} \in \{L, H\}^2$ will be exactly the same. This is so, because the state next period depends on the actual type of the player, not the reported type.

Let $\Delta_{k,t} := V_{k,t} - V_{k-1,t}$ for all k and t. Thus, depending of whether the type of player 2 was H or L in the previous period, the continuation payoff of player 1 of type L decreases in expectation by $p\Delta_{k,(L,H)} + (1-p)\Delta_{k+1,(L,L)}$ or $(1-p)\Delta_{k,(L,H)} + p\Delta_{k+1,(L,L)}$ when reporting truthfully compared to lying about her type. And player 1 of type H gains by reporting truthfully $p\Delta_{k,(H,H)} + (1-p)\Delta_{k+1,(H,L)}$ or $(1-p)\Delta_{k,(H,H)} + p\Delta_{k+1,(H,L)}$. In turn, the player of type L gains expression (6) for $t_1 = L$. Player 1 of type H loses expression (6) for $t_1 = H$.

Proposition 9 *For all $k = 1, ..., 2n-1$ and t, player 1 has incentives to report her type honestly.*

Proof. We will first establish the relations between different Δ's. For $k = 2, ..., 2n-1$, by applying (10)-(13) we obtain:

$$\Delta_{k,L,H} = \delta \left\{ p^2 \Delta_{k-1,L,H} + p(1-p)\Delta_{k,L,L} + (1-p)p\Delta_{k,H,H} + (1-p)^2 \Delta_{k+1,H,L} \right\}, \quad (16)$$

$$\Delta_{k,L,L} = \delta\left\{p(1-p)\Delta_{k-1,L,H} + p^2\Delta_{k,L,L} + (1-p)^2\Delta_{k,H,H} + (1-p)p\Delta_{k+1,H,L}\right\}, \quad (17)$$

$$\Delta_{k,H,H} = \delta\left\{(1-p)p\Delta_{k-1,L,H} + (1-p)^2\Delta_{k,L,L} + p^2\Delta_{k,H,H} + p(1-p)\Delta_{k+1,H,L}\right\}, \quad (18)$$

and

$$\Delta_{k,H,L} = \delta\left\{(1-p)^2\Delta_{k-1,L,H} + (1-p)p\Delta_{k,L,L} + p(1-p)\Delta_{k,H,H} + p^2\Delta_{k+1,H,L}\right\}. \quad (19)$$

In turn, for $k = 1$ and $t = (L, H)$, and $k = 2n$ and $t = (H, L)$, we obtain

$$\Delta_{1,L,H} = (1-\delta)A + \delta\left\{-p^2\Delta_{1,L,H} + (1-p)^2\Delta_{2,H,L}\right\} \quad (20)$$

and

$$\Delta_{2n,H,L} = (1-\delta)B + \delta\left\{(1-p)^2\Delta_{2n-1,L,H} - p^2\Delta_{2n,H,L}\right\}. \quad (21)$$

Finally, we must also introduce the terms $\Delta_{1,H,H}$ and $\Delta_{2n,L,L}$ because in states with $k = 1$ chip player 1 of type H may deviate by reporting L, and in states with $k = 2n - 1$ chips player 1 of type L may deviate by reporting H. This will result in moving to a state with 0 or $2n$ chips, respectively, but at profile $t = (H, H)$ or (L, L). We have that

$$\Delta_{1,H,H} = \delta\{-p(1-p)\Delta_{1,L,H} + p(1-p)\Delta_{2,H,L}\} + (1-\delta)\cdot A' \quad (22)$$

and

$$\Delta_{2n,L,L} = \delta\{p(1-p)\Delta_{2n-1,L,H} - p(1-p)\Delta_{2n,H,L}\} + (1-\delta)\cdot B'. \quad (23)$$

For $\delta = 1$, this system of linear equations is satisfied by all Δ's being equal to 0. For $\delta < 1$, we will evaluate Δ's in approximation by referring to the Implicit Function Theorem. By this theorem, one can differentiate the equations for Δ's with respect to δ, plug in $\delta = 1$ and $\Delta_{k,t} = 0$ for all k and t, and obtain a system of equations for the derivatives of Δ's. That is, if we replace each $\Delta_{k,t}$ by its derivative $\partial\Delta_{k,t}/\partial\delta$, then our system of linear equations must be satisfied for $\delta = 1$, and $(1-\delta)A$ and $(1-\delta)B$ replaced with $-A$ and $-B$, respectively. We will solve this new system of linear equation. The solution will be unique, which will also validate the use of the Implicit Function Theorem, as well as guarantee that all Δ's being equal to 0 is a unique solution of the system for $\delta = 1$.

We will show that all incentive constraints are satisfied for every given n, provided the discount factor is large enough.[18] To show this, we will derive a simple recursive condition which shows that the values of $\partial\Delta_{k,t}/\partial\delta$ evolve linearly with the current number of chips k. Then by using equations for $\partial\Delta_{1,t}/\partial\delta$ and $\partial\Delta_{2n,t}/\partial\delta$, we will find the values of all $\partial\Delta_{k,t}/\partial\delta$'s at $\delta = 1$. This will enable us to evaluate $\Delta_{k,t} \approx (1-\delta)\cdot(-\partial\Delta_{k,t}/\partial\delta)$.

In order to derive recursive condition for $\partial\Delta_{k,t}/\partial\delta$'s, add together the equations for $\partial\Delta_{k,t}/\partial\delta$'s corresponding to (16)-(19). This yields:

$$\partial\Delta_{k,L,H}/\partial\delta + \partial\Delta_{k,H,L}/\partial\delta = \partial\Delta_{k-1,L,H}/\partial\delta + \partial\Delta_{k+1,H,L}/\partial\delta.$$

This equation can also be expressed as $\partial\Delta_{k,H,L}/\partial\delta - \partial\Delta_{k-1,L,H}/\partial\delta = \partial\Delta_{k+1,H,L}/\partial\delta - \partial\Delta_{k,L,H}/\partial\delta$, which means that the value of $\partial\Delta_{k,H,L}/\partial\delta - \partial\Delta_{k-1,L,H}/\partial\delta$ is the same for all k. To simplify analysis, denote this value as ρ.

Subtracting the equations for $\partial\Delta_{k,t}/\partial\delta$'s corresponding to (16) from (19), we obtain that

$$\partial\Delta_{k,H,L}/\partial\delta - \partial\Delta_{k,L,H}/\partial\delta = \frac{2p-1}{p}\rho,$$

and from that we derive that

$$\partial\Delta_{k+1,H,L}/\partial\delta - \partial\Delta_{k,H,L}/\partial\delta = \partial\Delta_{k+1,L,H}/\partial\delta - \partial\Delta_{k,L,H}/\partial\delta = \frac{1-p}{p}\rho. \tag{24}$$

The last equation shows that the difference between $\partial\Delta_{k,t}/\partial\delta$'s for two consecutive values of k, given type profiles H,L or L,H, is independent of k. The difference between $\partial\Delta_{k,t}/\partial\delta$'s for two consecutive values of k, given type profiles L,L or H,H is equal to that given H,L or L,H by the equations corresponding to (17) or (18) for $\partial\Delta_{k,t}/\partial\delta$'s. In particular, this means that the values of $\partial\Delta_{k,t}/\partial\delta$'s change linearly with k.

We can now find the values of $\partial\Delta_{k,t}/\partial\delta$'s explicitly by using the equations for the states with extreme numbers of chips $k=1$ and $2n$. By the equation corresponding to (20),

$$\partial\Delta_{1,L,H}/\partial\delta = -A + \left\{-p^2\partial\Delta_{1,L,H}/\partial\delta + (1-p)^2\partial\Delta_{2,H,L}/\partial\delta\right\},$$

[18] For larger values of n, the threshold for the discount factor above which the incentive constraints are satisfied is typically larger.

and since $\partial\Delta_{2,H,L}/\partial\delta - \partial\Delta_{1,L,H}/\partial\delta = \rho$, this is equivalent to

$$\partial\Delta_{1,L,H}/\partial\delta = \frac{-A + (1-p)^2\rho}{2p}. \tag{25}$$

Similarly by the equation corresponding to (21) and $\partial\Delta_{2n,H,L}/\partial\delta - \partial\Delta_{2n-1,L,H}/\partial\delta = \rho$, we obtain that

$$\partial\Delta_{2n,H,L}/\partial\delta = \frac{-B - \rho(1-p)^2}{2p}. \tag{26}$$

Applying $2n-2$ times equation (24), we have that

$$\partial\Delta_{2n-1,L,H}/\partial\delta = \frac{-A + (1-p)^2\rho}{2p} + (2n-2)\frac{1-p}{p}\rho$$

and

$$\partial\Delta_{2n,H,L}/\partial\delta = \frac{-A + (1-p)^2\rho}{2p} + (2n-2)\frac{1-p}{p}\rho + \rho. \tag{27}$$

Combining equations (27) and (26), we obtain that

$$\rho = \frac{(A-B)/2p}{\frac{(1-p)^2}{p} + 1 + (2n-2)\frac{1-p}{p}}$$

Using this value of ρ, one can now find all $\partial\Delta_{k,t}/\partial\delta$'s for types H, L and L, H. Similarly, one can find the (approximate) values of $\Delta_{k,t}$. In particular, we can immediately see that $\frac{\Delta_{k,t}}{1-\delta}$'s for $k = 1, ..., 2n$ and types H, L and L, H are weighted averages of $\frac{A}{2p}$ and $\frac{B}{2p}$. And by (17) or (18), so are $\frac{\Delta_{k,t}}{1-\delta}$'s for $k = 2, ..., 2n-1$ and types L, L and H, H. The values of $\frac{\Delta_{k,t}}{1-\delta}$ for $k = 1$ are the closest to $\frac{A}{2p}$, and the values of $\frac{\Delta_{k,t}}{1-\delta}$ for $k = 2n$ are the closest to $\frac{B}{2p}$. The higher k is, the closer the value $\frac{\Delta_{k,t}}{1-\delta}$ is to $\frac{B}{2p}$. By Assumption I from Section 3, this shows that player 1 has incentives to report her type truthfully, except two cases: (i) when her type is L and she has $2n-1$ chips; (ii) when her type is H and she has 1 chip. These two cases are exceptional because the analysis of incentives involves $\Delta_{2n,(L,L)}$ and $\Delta_{1,(H,H)}$, respectively. So, they must be considered separately.

Consider case (i). By deviating and reporting H, player 1 gains, compared to reporting truthfully, $p\Delta_{2n-1,(L,H)} + (1-p)\Delta_{2n,(L,L)}$ or $(1-p)\Delta_{2n-1,(L,H)} + p\Delta_{2n,(L,L)}$, but loses expression (6) for $t_1 = L$. By Assumption I, $\Delta_{2n-1,(L,H)}$ is smaller than the loss, and by Assumption II, $\Delta_{2n,(L,L)}$ is smaller than the loss, preventing player 1 from deviating. Case (ii) follows from analogous arguments. ■

10 Appendix B

Denote by V_k the continuation payoff of player 1 with k chips. Then, V_k, $0 < k < 2n$, can be written as:

$$V_k = \frac{1}{3}\left[1(1-\delta)\frac{1}{6} + \delta\left(\frac{1}{3}V_k + \frac{1}{3}\left[p_{1,2}V_{k+1} + (1-p_{1,2})V_k)\right] + \frac{1}{3}\left[p_{1,3}V_{k+1} + (1-p_{1,3})V_k)\right]\right)\right] +$$

$$+\frac{1}{3}\left[2(1-\delta)\frac{3}{6} + \delta\left(\frac{1}{3}\left[p_{1,2}V_{k-1} + (1-p_{1,2})V_k)\right] + \frac{1}{3}V_k + \frac{1}{3}\left[p_{2,3}V_{k+1} + (1-p_{2,3})V_k)\right]\right)\right] +$$

$$+\frac{1}{3}\left[3(1-\delta)\frac{5}{6} + \delta\left(\frac{1}{3}\left[p_{1,3}V_{k-1} + (1-p_{1,3})V_k)\right] + \frac{1}{3}\left[p_{2,3}V_{k-1} + (1-p_{2,3})V_k)\right] + \frac{1}{3}V_k\right)\right].$$

The value function V_k is represented as the sum of three terms, each term corresponds to one of the three types 1, 2, and 3. For example, the first of these terms represents the case when the type of player 1 is equal to 1. In this case, player 1 values the apple at 1, and obtains it with probability 1/6. With probability 1/3 player 2 is also of type 1, and therefore number of chips remains the same. With probability 1/3 player 2's type is 2, and then player 1 obtains a chip with probability $p_{1,2}$. Finally, with probability 1/3 player 2's type is 3, and then player 1 obtains a chip with probability $p_{1,3}$.

The expression above can be simplified to:

$$V_k = (1-\delta)\frac{22}{18} + \delta[pV_{k-1} + (1-2p)V_k + pV_{k+1}], \tag{28}$$

where

$$p = \frac{1}{9}(p_{1,2} + p_{1,3} + p_{2,3}).$$

For $k = 2n$, one has

$$V_{2n} = (1-\delta)2 + \delta[p_r V_{2n-1} + (1-p_r)V_{2n}].$$

Indeed, if a player has all chips, she obtains the apple, no matter what the types are, which has the expected payoff of 2. With probability p_r the opponent obtains a chip back, and with the remaining probability the distribution of chips remains the same.

When δ goes to 1,

$$V_{2n} = \frac{(1-\delta)2 + \delta p_r V_{2n-1}}{(1-\delta)(1-p_r) + p_r} = \frac{(1-\delta)2}{p_r} + \left[1 - \frac{(1-\delta)}{p_r}\right] V_{2n-1} + o(1-\delta), \tag{29}$$

where $o(1-\delta)$ stands for an expression that goes to 0 (when δ goes to 1) faster than $1-\delta$. To check the second equality, multiply it (omitting term $o(1-\delta)$) by $(1-\delta)(1-p_r)+p_r$, and remove each term containing $(1-\delta)^2$. Similarly,

$$V_0 = \left[1 - \frac{(1-\delta)}{p_r}\right] V_1 + o(1-\delta). \tag{30}$$

From now on, we will omit all terms $o(1-\delta)$; That is, all equalities and equations should be understood as holding up to such a term.

We will now derive the following recursive formula:

$$V_k = (1-\delta)\alpha_k + [1 - (1-\delta)\beta_k]V_{k+1} \tag{31}$$

for some coefficients α_k and β_k; these coefficients will also be determined. By (30), this formula holds for $k = 0$ with $\alpha_0 = 0$ and $\beta_0 = 1/p_r$. Assume that it holds for k. Plug the expression for V_k from (31) into equation (28) for $k+1$ to compute that

$$V_{k+1} = (1-\delta)\left(\frac{22}{18p} + \alpha_k\right) + \left[1 - (1-\delta)\left(\frac{1}{p} + \beta_k\right)\right] V_{k+2},$$

which gives the recursive equations:

$$\alpha_{k+1} = \alpha_k + \frac{22}{18p},$$

$$\beta_{k+1} = \beta_k + \frac{1}{p}.$$

Since $\alpha_0 = 0$ and $\beta_0 = 1/p_r$,

$$\alpha_{2n-1} = (2n-1)\frac{22}{18p},$$

and

$$\beta_{2n-1} = \frac{1}{p_r} + (2n-1)\frac{1}{p}.$$

By using (31) for $k = 2n - 1$ and (29), compute that

$$V_{2n} = \frac{2s + (2n-1)\frac{22}{18}}{2s + (2n-1)}, \tag{32}$$

where

$$s = \frac{p}{p_r}.$$

Thus, V_{2n} is a weighted average of 22/18 - the efficient per-player payoff, and 1 - the expected per-player payoff in the inefficient state (i.e., when the player with $2n$ chips obtains on average $(1/3)1 + (1/3)2 + (1/3)3 = 2$, and the opponent obtains 0).

To approximate the efficient outcome s must be much lower than $2n$. We will show that this implies that incentive constraints cannot be satisfied for all k. First, estimate $V_{k+1} - V_k$ - the value of an extra chip for a player with k chips. By (31),

$$V_{k+1} - V_k = -(1-\delta)\alpha_k + (1-\delta)\beta_k V_{k+1}.$$

It also follows from (31) that V_{k+1}, for $k = 0, ..., 2n-1$, is equal to (up to a term $o(1-\delta)$) the same weighted average as V_{2n} in (32). By plugging in this weighted average for V_{k+1}, and plugging in the formulas for α_k, β_k, we obtain that

$$V_{k+1} - V_k = -(1-\delta)k\frac{22}{18p} + (1-\delta)\left[\frac{1}{p_r} + k\frac{1}{p}\right]\frac{2s + (2n-1)\frac{22}{18}}{2s + (2n-1)}.$$

Thus, $V_{k+1} - V_k$ is linear in k. At $k = 0$, one has

$$V_1 - V_0 = (1-\delta)\frac{1}{p_r}\frac{2s + (2n-1)\frac{22}{18}}{2s + (2n-1)}. \tag{33}$$

In turn at $k = 2n - 1$, one has

$$V_{2n} - V_{2n-1} = -(1-\delta)(2n-1)\frac{22}{18sp_r} + (1-\delta)\left[\frac{1}{p_r} + (2n-1)\frac{1}{sp_r}\right]\frac{2s + (2n-1)\frac{22}{18}}{2s + (2n-1)}. \tag{34}$$

This enables us to compute the ratio of $V_1 - V_0$ to $V_{2n} - V_{2n-1}$, and using the fact that $2n - 1$ must be much larger than s, we obtain that this ratio is close to 11/7.

Player 1 with type t_i by reporting type 1 obtains the flow payoff of

$$(1-\delta)t_i\frac{1}{6};$$

and if the current continuation payoff is V_k, this report will change her continuation payoff by

$$\left[\frac{1}{3}p_{1,2}+\frac{1}{3}p_{1,3}\right](V_{k+1}-V_k).$$

We have omitted the factor δ, because $V_{k+1}-V_k$ is of order $1-\delta$, and all our quantities are evaluated up to terms $o(1-\delta)$.

Therefore, the total effect on payoff of reporting type 1 is

$$(1-\delta)t_i\frac{1}{6}+\left[\frac{1}{3}p_{1,2}+\frac{1}{3}p_{1,3}\right](V_{k+1}-V_k). \tag{35}$$

Similarly, the effect on player 1's payoff of reporting type 2 is:

$$(1-\delta)t_i\frac{3}{6}+\frac{1}{3}p_{1,2}(V_{k-1}-V_k)+\frac{1}{3}p_{2,3}(V_{k+1}-V_k), \tag{36}$$

and the effect on player 1's payoff of reporting type 3 is:

$$(1-\delta)t_i\frac{5}{6}+\left[\frac{1}{3}p_{1,3}+\frac{1}{3}p_{2,3}\right](V_{k-1}-V_k). \tag{37}$$

The strategies are incentive compatible when for every type $t_i\in\{1,2,3\}$ player 1 has incentives to report this type honestly; in particular, she prefers to report this type type to reporting any neighbor type.[19] Types $1,2$ prefer not to mimic one another if:

$$(1-\delta)\frac{2}{6}<\frac{1}{3}(p_{1,2}+p_{1,3}-p_{2,3})(V_{k+1}-V_k)+\frac{1}{3}p_{1,2}(V_k-V_{k-1})<(1-\delta)\frac{4}{6}; \tag{38}$$

similarly, types $2,3$ prefer not to mimic one another if:

$$(1-\delta)\frac{4}{6}<\frac{1}{3}p_{2,3}(V_{k+1}-V_k)+\frac{1}{3}(-p_{1,2}+p_{1,3}+p_{2,3})(V_k-V_{k-1})<(1-\delta)\frac{6}{6}. \tag{39}$$

[19]In fact, this condition is equivalent to incentive compatibility, but this will be inessential for our analysis.

Indeed, we obtain these inequalities from (35)-(37).

Since the middle terms in these inequalities are linear in $V_k - V_{k-1}$, which is linear in k, and the ratio $(V_1 - V_0)/(V_{2n} - V_{2n-1})$ is close to $11/7$, the ratio of the two bounds: $(1-\delta)\frac{4}{6}$ and $(1-\delta)\frac{2}{6}$ in (38), and $(1-\delta)\frac{6}{6}$ and $(1-\delta)\frac{4}{6}$ in (39) has to be at least $11/7$. It is the case for (38) but not for (39). This means that the strategies are not incentive compatible.

Remark 1 *The ratio of two neighbor types* $3/2$ *being lower than the ratio* $(V_1 - V_0)/(V_{2n} - V_{2n-1})$ *is what made the random chip strategies violating incentive constraints. Were it not the case, one could fairly easily find probabilities* $p_{1,2}$, $p_{1,3}$, $p_{2,3}$ *and* p_r *such that the strategies would be incentive compatible, and would attain an almost efficient outcome for large enough discount factors.*