Large Contests - Online Appendix

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We choose an equilibrium for each contest, and refer to the sequence in which the n-th element is the equilibrium of the n-th contest as the sequence of equilibria. For both our theorems, we will show that every subsequence of this sequence contains a further subsequence that satisfies the statement of the theorem. This suffices, because the following observation can be applied with Z_n being the set of equilibria of contest n.

(**Subsequence Property**) Given a sequence of sets $\{Z_n : n = 1, 2, \ldots\}$, suppose that for every subsequence $\{Z_{n_k} : k = 1, 2, \ldots\}$, every sequence $\{z_{n_k} : k = 1, 2, \ldots\}$ with $z_{n_k} \in Z_{n_k}$ contains a subsequence $\{z_{n_{k_l}} : l = 1, 2, ...\}$ such that every element $z_{n_{k_l}}$ has some property. Then there exists an N such that for every $n \geq N$ every element in Z_n has this property.[1]

1 Proof of Theorem 1

We begin with an outline of the proof. Given a subsequence of equilibria, each equilibrium in the subsequence induces for each player a mapping from bids to expected percentile rankings. We consider the average of those mappings, and G^{-1} composed with this average gives a mapping T^n from bids to prizes. As n increases, this mapping approximates the equilibrium mappings from bids to prizes of all players in the n-th contest. We then use Helly's (1912) selection theorem to find a subsequence of T^n that converges to some limit mapping T from bids to prizes. We show that T is continuous and the subsequence of T^n converges uniformly to T. Then, for each agent type we define the set of optimal bids when T is treated as an

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[1] Otherwise, there would be a sequence $\{z_{n_k} : k = 1, 2, \ldots\}$ with $z_{n_k} \in Z_{n_k}$ of elements without the property.

inverse tariff, and define the correspondence from agent types to sets of optimal bids. We consider a small neighborhood of the graph of this correspondence, and show that for large n every player i's best responses in the n-th contest are in the "x_i^n-slice" of this neighborhood. Such a slice could in principle be large even if the set of optimal bids of the corresponding agent type is small (this would happen if the set of optimal bids of a nearby agent type is large). We show, however, that under strict single crossing each agent type has a single optimal bid, which is continuous and weakly increases in the agent type. This implies that every player's best response set, and therefore the support of her equilibrium strategy, is bounded within an arbitrarily small interval as n increases. We then conclude that the unique mechanism induced by T implements the assortative allocation. This demonstrates part (b) in the statement of the theorem; part (a) then follows easily.

For the proof, we take the subsequence of equilibria to be the sequence of equilibria (this simplifies notation and has no effect on the proofs). We denote the equilibrium of the n-th contest by $\sigma^n = (\sigma_1^n, \ldots, \sigma_n^n)$, where σ_i^n is player i's equilibrium strategy; a strategy of player i is a random variable taking values in $X \times B$ whose marginal distribution on X coincides with the distribution of player i's types F_i^n. By referring to player i bidding with some probability in a subset S of B, we mean the probability of the set $X \times S$, i.e., the probability of S measured by the marginal distribution of player i's strategy on B.

We denote by $R_i^n(t)$ the random variable that is the percentile location of player i in the ordinal ranking of the players in the n-th contest if she bids slightly above t and the other players employ their equilibrium strategies.[2] That is,

$$R_i^n(t) = \frac{1}{n}\left(1 + \sum_{k \neq i} 1_{\{\sigma_k^n \in X \times [0,t]\}}\right),$$

where $1_{\{\sigma \in X \times [0,t]\}}$ is 1 if $\sigma \in X \times [0,t]$ and 0 otherwise. Let

$$A_i^n(t) = \frac{1}{n}\left(1 + \sum_{k \neq i} \Pr\left(\sigma_k^n \in X \times [0,t]\right)\right)$$

be the expected percentile ranking of player i. Then, by Hoeffding's inequality, for all t in B we have

$$\Pr\left(\left|R_i^n(t) - A_i^n(t)\right| > \delta\right) < 2\exp\left\{-2\delta^2(n-1)\right\}. \tag{1}$$

Finally, let

$$A^n(t) = \frac{1}{n}\sum_{i=1}^{n} A_i^n(t)$$

[2] This is the infimum of her ranking if she bids above t, which is equivalent to bidding t and winning any ties there. If ties happen with probability 0, then this is equivalent to bidding t.

be the average of the expected percentiles rankings of the players in the n-th contest if they bid t and the other players employ their equilibrium strategies.

Let T^n be the mapping from bids to prizes induced by A^n. That is, $T^n(t) = (G^n)^{-1}(A^n(t))$, where $(G^n)^{-1}(z) = \inf\{y : G^n(y) \geq z\}$ for $z > 0$, and $(G^n)^{-1}(0) = \inf\{y : G^n(y) > 0\}$. (In words, $(G^n)^{-1}(z)$ is the prize of an agent with percentile ranking z when prizes are distributed according to G^n.) Since every T^n is (weakly) increasing, by Helly's (1912) selection theorem for monotone functions the sequence T^n contains a subsequence that converges pointwise to a function $T : B \to Y$. For the rest of the proof, denote this subsequence by T^n.

We first describe some properties of inverse tariff T:

(1) T is (weakly) increasing, because every T^n is (weakly) increasing.
(2) $T(0) = 0$, otherwise players bidding 0 would have profitable deviations.[3]
(3) $T(b_{\max}) = 1$, because $A^n(b_{\max}) = 1$ and therefore $T^n(b_{\max}) = 1$.

In addition, we will use the following property of discrete contest equilibria:

(No-Gap Property) In any equilibrium, there is no interval $(a, b) \in B$ of positive length in which all players bid with probability 0 and some player bids in $[b, b_{\max}]$ with positive probability.

Proof: Suppose the contrary, and consider such a maximal interval (a, b). A player would only bid b or slightly higher than b if some other player bids b with positive probability. But the player who bids b with positive probability would be better off either by slightly increasing her bid (if another player bids b and winning the tie leads to a higher prize) or by decreasing her bid (in the complementary case).

Our first lemma shows that T is continuous. (In order not to obscure the structure of the proof, we relegate to the end of the section the proofs of all lemmas.)

Lemma 1 *For any $t \in B$ and any sequences $q^m \uparrow t$ and $r^m \downarrow t$ in B, we have* $\lim T(q^m) = \lim T(r^m) = T(t)$.

The idea of the proof is that if T were discontinuous at some t, then for large n it would be better to bid slightly above t than slightly below t. But if no player bids slightly below t, then by the No-Gap Property no player bids t or above.

[3] Indeed, suppose to the contrary that $T(0) > 0$. This means that for some $\delta > 0$ and large enough n, $A^n(0) > G^n(0) + \delta$. Thus, a fraction of at least $G^n(0) + \delta$ players bid 0 in the n-th contest with positive probability. Any one of them would be better off bidding slightly above 0, and winning against all other players who bid 0, than bidding 0 and with positive probability losing to all other players who bid 0.

Continuity and monotonicity of T imply the following result.

Lemma 2 *T^n converges to T uniformly on B.*

We now relate the inverse tariff T to players' behavior in the equilibria that correspond to the sequence T^n. Denote by BR_x type x's set of optimal bids given T, i.e., the bids t that maximize $U(x, T(t), t)$. Denote by $BR(\varepsilon)$ the ε-neighborhood of the graph of the correspondence that assigns to every type x the set BR_x.[4] Denote by $BR_x(\varepsilon)$ the set of bids t such that $(x, t) \in BR(\varepsilon)$.

Note that $BR(\varepsilon)$ is a 2-dimensional open set, while each $BR_x(\varepsilon)$ is a 1-dimensional "slice" of $BR(\varepsilon)$. Using sets $BR_x(\varepsilon)$, we can characterize players' equilibrium behavior.

Lemma 3 *For every $\varepsilon > 0$, there is an N such that for every $n \geq N$, in the equilibrium of the n-th contest every best response of every type x_i^n of every player i belongs to $BR_{x_i^n}(\varepsilon)$.*

Strict single crossing implies several properties of BR_x.

Lemma 4 *For every x the set BR_x is a singleton. In addition, the function br that assigns to x the single element of BR_x is continuous and weakly increasing.*

Lemma 4 implies that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $BR_x(\delta) \subseteq [br(x) - \varepsilon, br(x) + \varepsilon]$ for every type x. We therefore have the following corollary of Lemmas 3 and 4.

Corollary 1 *For every $\varepsilon > 0$, there is an N such that for every $n \geq N$, in the equilibrium of the n-th contest every best response of every type x_i^n of every player i belongs to $(br(x_i^n) - \varepsilon, br(x_i^n) + \varepsilon)$.*

To prove part (b) of the theorem we need to show that $T \circ br$ is the assortative allocation. This is done by the following lemma.

Lemma 5 *$G^{-1}(F(x)) = T(br(x))$ for all types x.*

Thus, the mechanism that prescribes for type x prize $T(br(x))$ and bid $br(x)$ is a tariff mechanism that implements the assortative allocation. Moreover, every type can get at least 0 by bidding 0, and type 0 gets no more than 0 (because $T(br(0)) = 0$).

To complete the proof, it remains to show part (a) of the theorem. In short, this part follows from Corollary 1 and Hoeffding's inequality (see Appendix 1.6).

[4] That is, $BR(\varepsilon)$ is the union over all types x and bids $t \in BR_x$ of the open balls of radius ε centered at (x, t).

1.1 Proof of Lemma 1

Suppose first the lemma is false for some $t \in (0, b_{\max}]$ and $q^m \uparrow t$. Let $y' = \lim T(q^m)$ and $y'' = T(t)$ (the limit exists by the monotonicity of T), and let $\gamma = (y'' - y')/2$.

Suppose first that $U(0, y, t)$ strictly increases in y. (Recall that we assumed $U(x, y, t)$ strictly increases in y only for $x > 0$.) Then, by uniform continuity of U, there exist $\delta, \Delta > 0$ such that every type x gains at least Δ from obtaining a prize higher by γ at a bid higher by δ. More precisely,

$$U(x, y + \gamma, t + \delta) - U(x, y, t) \geq \Delta \tag{2}$$

for all x, y, and t such that $y + \gamma$ and $t + \delta$ belong to the domain of U. This implies, as U is bounded, that every type x strictly prefers bidding $t + \delta$ and obtaining with sufficiently high probability a prize sufficiently close to $y + \gamma$ to bidding t and obtaining with sufficiently high probability a prize sufficiently close to y, independently of the prizes obtained with the remaining probability.

Choose $t' = q^m$ such that $t - t' < \delta$. Next, choose n large enough so that $|T^n(t) - T(t)| < \gamma/2$ and $|T^n(t') - T(t')| < \gamma/2$. This implies that $T^n(t) - T^n(s) > \gamma$ for any bid $s \leq t'$.

By choosing n large enough, we guarantee (see (1), which applies uniformly to all bids) that $R_i^n(s)$, the percentile ranking of player i who bids s in the n-th contest, is close to $A^n(s)$ with high probability, and $R_i^n(t)$ is close to $A^n(t)$ with high probability. Thus, every type x obtains a prize sufficiently close to $T^n(t)$ with a sufficiently high probability by bidding (slightly above) t, and obtains a prize that is with a sufficiently high probability at most slightly higher than $T^n(t')$ by bidding (slightly above) any $s \leq t'$.[5] Therefore, because $t - t' < \delta$**,** no player bids any $s \in (t - \delta, t']$ with positive probability, so by the No-Gap Property $T^n(t') = 1$. But $T^n(t') \to T(t') \leq y' < y'' \leq 1$, a contradiction.

When $U(0, y, t)$ only weakly increases in y, the argument above shows that for any $\varepsilon > 0$ there exist $\delta, \Delta > 0$ for which (2) holds for every type $x \in [\varepsilon, 1]$. There also exists $t' = q^m$ such that $t - t' < \delta$ and $T^n(t) - T^n(t') > 3\gamma/2$ for large enough n. Letting $t'' = \inf\{s : T^n(s) \geq T^n(t) - \gamma\} \in (t', t]$ we see that only players with types lower than ε can bid in $[t', t'')$. Thus, for small enough ε (by continuity of G^{-1} and convergence of $(G^n)^{-1}$ to G^{-1}), in order to increase $T^n(t')$ to $T^n(t') + \gamma/2$ multiple players with types ε or higher must bid t'' with positive probability and therefore tie there. But then any one of these players could profitably deviate to bidding slightly above t''.[6]

[5] For $t = b_{\max}$, bidding "slightly above $b_{\max}$" is impossible. But by bidding $b_{\max}$ a player wins with probability 1, because $b_{\max}$ is strictly dominated by 0 for all players.

[6] By doing so such a player would obtain with high probability a prize of at least $T^n(t') + \gamma/2$ instead of losing the tie with positive probability and then obtaining with high probability a prize of at most $T^n(t') + \varepsilon$.

The argument is analogous if we suppose that the lemma is false for some $t \in (0, b_{\max})$ and $r^m \downarrow t$. If $t = 0$, then the above proof shows that for large n no player bids $t = 0$ with positive probability. This means, in turn, that sufficiently small bids give lower payoffs than $t' = r^m$ such that $t' - t < \delta$. Thus, no player bids close to $t = 0$ with positive probability, which contradicts the No-Gap Property.

1.2 Proof of Lemma 2

Suppose the contrary. Then, there is some $\delta > 0$ and a sequence of integers $n_1, n_2, \ldots$ such that for every n_k there is some bid t_k with $|T^{n_k}(t_k) - T(t_k)| > \delta$. Passing to a subsequence if necessary, we assume that $t_k \to t$.

Consider numbers q' and q'' such that $q' < t < q''$ and $T(q'') - T(q') < \delta/2$; such numbers exist because T is continuous.[7] For large enough values of k, we have that $|T^{n_k}(q') - T(q')| < \delta/2$ and $|T^{n_k}(q'') - T(q'')| < \delta/2$.

For any $t' \in [q', q'']$, either (a) $T^{n_k}(t') \geq T(t')$, or (b) $T^{n_k}(t') \leq T(t')$.

By monotonicity of T and T^{n_k}, we have

$$T^{n_k}(t') - T(t') \leq T^{n_k}(q'') - T(q') \leq |T^{n_k}(q'') - T(q'')| + |T(q'') - T(q')| < \delta$$

in case (a), and

$$T(t') - T^{n_k}(t') \leq T(q'') - T^{n_k}(q') \leq |T(q'') - T(q')| + |T(q') - T^{n_k}(q')| < \delta$$

in case (b).

Since $t_k \in [q', q'']$ for large enough k, we obtain a contradiction to the assumption that $|T^{n_k}(t_k) - T(t_k)| > \delta$ for all such k.

1.3 Proof of Lemma 3

Suppose to the contrary that for arbitrarily large n, in the equilibrium of the n-th contest some type x_i^n of some player i has a best response that belongs to the complement of $BR_{x_i^n}(\varepsilon)$. Passing to a convergent subsequence if necessary, we assume that $x_i^n \to x^*$.

Note that for every x there is a $\delta_x > 0$ such that (under the inverse tariff) any bid from the complement of $BR_x(\varepsilon)$ gives type x a payoff lower by at least δ_x than any element of BR_x does. Let $\delta = \delta_{x^*}$.

We have that:

[7] If $t = 0$ set $q' = 0$, and if $t = b_{\max}$ set $q'' = b_{\max}$.

1. The maximal payoff of type x, attained at any bid from BR_x, is continuous in x.

This follows from Berge's Theorem.

2. For every $\rho > 0$, for sufficiently large n the highest payoff that type x_i^n can obtain by bidding in the complement of $BR_{x_i^n}(\varepsilon)$ cannot exceed by ρ the highest payoff that type x^* can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$.

Indeed, suppose that for a sequence n_k diverging to ∞ type $x_i^{n_k}$ obtains by bidding some t_k in the complement of $BR_{x_i^{n_k}}(\varepsilon)$ a payoff at least ρ higher than the highest payoff that type x^* can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$. Passing to a convergent subsequence if necessary, we assume that $t_k \to t$. Since every $(x_i^{n_k}, t_k)$ belongs to the complement of $BR(\varepsilon)$, so does (x^*, t); thus, (x^*, t) belongs to the complement of $BR_{x^*}(\varepsilon)$. However, by continuity of the payoff functions, bidding t gives type x^* a payoff by at least ρ higher than the highest payoff that type x^* can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$, a contradiction.

By 1 and 2, for sufficiently large n, any bid in the complement of $BR_{x_i^n}(\varepsilon)$ gives type x_i^n a payoff lower by at least $\delta/2$ than any bid in $BR_{x_i^n}$. Indeed, by 2 applied to $\rho = \delta/4$, any bid in the complement of $BR_{x_i^n}(\varepsilon)$ gives type x_i^n a payoff at most $\delta/4$ higher than the highest payoff that type x^* can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$. This last payoff is in turn lower than the payoff that type x^* obtains by bidding in BR_{x^*} by at least δ. And by 1, the payoff that type x_i^n obtains by bidding in $BR_{x_i^n}$ cannot be lower by more than $\delta/4$ than the payoff that type x^* obtains by bidding in BR_{x^*}.

By uniform convergence of T^n to T, the analogous statement, with $\delta/2$ replaced with some smaller positive number and T replaced with T^n, is also true. This means, however, that for sufficiently large n, player i would be strictly better off bidding slightly above any bid in $BR_{x_i^n}$ when her type is x_i^n than bidding in the complement of $BR_{x_i^n}(\varepsilon)$. This is because (1) implies that for sufficiently large n, by bidding slightly above t the player obtains a prize arbitrarily close to $T^n(t)$ with probability arbitrarily close to 1.

1.4 Proof of Lemma 4

Observe that for any $x' < x''$, strict single crossing implies that if $t' \in BR_{x'}$ and $t'' \in BR_{x''}$, then $t' \le t''$. Suppose that $BR_{x'}$ contained two bids, $t_1 < t_2$, for some type x. The first observation and Lemma 3 imply that for any $0 < \varepsilon < (t_2 - t_1)/4$, for sufficiently large n only players with types in $I = [\max\{x' - \varepsilon, 0\}, \min\{x' + \varepsilon, 1\}]$ may bid in the interval $[t_1 + (t_2 - t_1)/4, t_2 - (t_2 - t_1)/4]$.

Consider obtaining a prize that is Δ higher in the limit prize distribution[8] by increasing

[8] More precisely, given an initial prize y', the prize that is Δ higher in the limit prize distribution is the prize y'' such that $\Delta = G(y'') - G(y')$. The prize that is Δ higher in the prize distribution G^n is defined

the bid from $t_1 + (t_2 - t_1)/4$ to $t_2 - (t_2 - t_1)/2$. If Δ is sufficiently small, then by continuity of G^{-1} the increase in the prize is small as well, so the associated increment in utility is negative for all types, and uniformly bounded away from 0.

Therefore, taking $\Delta/2 = F(\min\{x' + \varepsilon, 1\}) - F(\max\{x' - \varepsilon, 0\})$, if $\varepsilon > 0$ is sufficiently small, then for sufficiently large n every type of every player is better off bidding $t_1 + (t_2 - t_1)/4$ than bidding $t_2 - (t_2 - t_1)/2$. This is because with high probability the higher bid leads to a prize that is approximately only $\Delta/2$ higher in the prize distribution G^n.[9] By convergence of $(G^n)^{-1}$ to $(G)^{-1}$, for sufficiently large n this prize is not much more than $\Delta/2$ higher in the limit prize distribution.

Moreover, every type of every player is better off bidding $t_1 + (t_2 - t_1)/4$ than bidding any bid in interval $(t_2 - (t_2 - t_1)/2, t_2 - (t_2 - t_1)/4)$, because such bids are even more costly than $t_2 - (t_2 - t_1)/2$, and enable a player to obtain a prize that is with high probability not much more than $\Delta/2$ higher in the limit prize distribution than the prize the player obtains by bidding $t_2 - (t_2 - t_1)/2$. Therefore, no player bids in the interval $((t_2 - (t_2 - t_1)/2, t_2 - (t_2 - t_1)/4))$ with positive probability, so by the No-Gap Property $T^n(t_2 - (t_2 - t_1)/4) = 1$ for sufficiently large n. Thus, $T(t_2 - (t_2 - t_1)/4) = 1$, so t_2 cannot be in BR_x, because bidding slightly above $t_2 - (t_2 - t_1)/4$ gives type x a higher payoff.

Consequently, BR_x is a singleton for any x, and by strict single crossing, br is weakly increasing. An argument analogous to the argument used to show that BR is a singleton also shows that br is continuous.[10]

1.5 Proof of Lemma 5

Consider an arbitrary type x. Let $x^{\min} = \min\{z : br(z) = br(x)\}$ and $x^{\max} = \max\{z : br(z) = br(x)\}$ ($x^{\min}$ and $x^{\max}$ are well defined because br is continuous).

First, observe that $G^{-1}(F(x^{\min})) = G^{-1}(F(x^{\max}))$. Indeed, by Corollary 1, for sufficiently large n all types in the interval $[x^{\min}, x^{\max}]$ bid in the n-th contest close to $br(x)$. Suppose that $G^{-1}(F(x^{\min})) < G^{-1}(F(x^{\max}))$, and consider the players whose type belongs

similarly.

[9]This follows from the convergence of F^n to F and Hoeffding's inequality applied to random variables

$$Z_i^n = \begin{cases} 1 \text{ if } \min\{x' + \varepsilon, 1\} \leq x_i^n \leq \max\{x' - \varepsilon, 0\}, \\ 0 \text{ otherwise,} \end{cases}$$

for $i = 1, ..., n$.

[10]More precisely, suppose that br is discontinuous at some x, and apply the argument to $t_1 = br(x_1)$ and $t_2 = br(x_2)$ where x_1 and x_2 are slightly lower and higher, respectively, than x.

to $[x^{\min}, x^{\max}]$ with positive probability.[11] Among these players, the one whose expected prize is the lowest contingent on having a type in this interval can profitably deviate to bidding slightly above $br(x)$, thereby outbidding the other players with a type in this interval and obtaining a discretely higher prize.

Suppose that $x^{\min} > 0$. By Corollary 1 for any $\delta > 0$, there is an N such that if $n \geq N$, then the equilibrium bids of every player with type lower than $x^{\min} - \delta$ are lower than $br(x^{\min})$, and the equilibrium bids of every player with type higher than $x^{\min}$ are higher than $br\left(x^{\min} - \delta\right)$. Therefore, a player who bids $br\left(x^{\min}\right)$ outbids all players with types lower than $x^{\min} - \delta$, so $T^n(br\left(x^{\min}\right)) \geq (G^n)^{-1}(F^n\left(x^{\min} - \delta\right))$, and a player who bids $br\left(x^{\min} - \delta\right)$ is outbid by all players with types higher than $x^{\min}$, so $T^n\left(br\left(x^{\min} - \delta\right)\right) \leq (G^n)^{-1}(F^n\left(x^{\min}\right))$. Since T^n converges to T, T and br are continuous, $(G^n)^{-1}$ converges to (G^{-1}), F^n converges to F, and F and G^{-1} are continuous, we obtain $T\left(br\left(x^{\min}\right)\right) = G^{-1}\left(F\left(x^{\min}\right)\right)$.

Similarly, if $x^{\max} < 1$, we obtain that $T\left(br\left(x^{\max}\right)\right) = G^{-1}\left(F\left(x^{\max}\right)\right)$.

Thus, since $br(x) = br(x^{\min}) = br(x^{\max})$ and $G^{-1}(F(x)) = G^{-1}(F(x^{\min})) = G^{-1}(F(x^{\max}))$, we have that $T\left(br\left(x\right)\right) = G^{-1}\left(F\left(x\right)\right)$ when $x^{\min} > 0$ or $x^{\max} < 1$. Finally, it cannot be that $x^{\min} = 0$ and $x^{\max} = 1$, because $0 = G^{-1}(F\left(0\right)) < G^{-1}(F\left(1\right)) = 1$.

1.6 Proof of Part (a) of Theorem 1

Consider a type $x \in X$. Let $t = br\left(x\right)$, and let t' and t'' be such that $T\left(t'\right) = T(t) - \varepsilon/3$ and $T\left(t''\right) = T(t) + \varepsilon/3$. Finally, let x' and x'' be such that $t' = br\left(x'\right)$ and $t'' = br\left(x''\right)$. (Take $x' = 0$ and $t' = 0$ if $T(t) - \varepsilon/3 < 0$, and $x'' = 1$ and $t'' = br(1)$ if $T(t) + \varepsilon/3 > 1$.)

By Lemma 3, for sufficiently large n, every player with type no higher than x' bids less than the player with type x, and every player with type no lower than x'' bids more than the player with type x. By Hoeffding's inequality, the player of type x outbids with high probability at least a fraction of players close to $F^n\left(x'\right)$.

Since F^n converges to F, she outbids with high probability at least a fraction of players close to $F\left(x'\right)$. So, since $(G^n)^{-1}$ converges to G^{-1}, she obtains (with high probability) a prize no lower than $G^{-1}\left(F\left(x'\right)\right) - \varepsilon/3 = T(t) - 2\varepsilon/3$. Similarly, for sufficiently large n a player of type x outbids with high probability at most a fraction of players close to $F\left(x''\right)$, and so she obtains (with high probability) a prize no higher than $G^{-1}\left(F\left(x''\right)\right) = T\left(t\right) + 2\varepsilon/3$. (These bounds are immediate if $y - \varepsilon/2 < 0$ or if $y + \varepsilon/2 > 1$.) Thus, type x obtains (with high probability) a prize which differs from $G^{-1}(F(x))$ by at most $2\varepsilon/3$.

[11] For large n, at least a fraction of players close to $F\left(x^{\max}\right) - F\left(x^{\min}\right)$ have types that belong to $[x^{\min}, x^{\max}]$ with positive probability.

This proves part (a) for a single x, but we must show that there is an N such that for any $n \geq N$ part (a) holds for all x simultaneously. Such an N can be obtained by taking a finite grid of types x, and the corresponding grid of bids $br(x)$ such that $|T(t^1) - T(t^2)| < \varepsilon/3$ for any pair of neighboring elements t^1, t^2 of the grid, and taking the largest N among the N's corresponding to x's from the grid.

2 Proof of Theorem 2

Recall that $G^{-1}(z) = \inf\{y : G(y) \geq z\}$ for $z > 0$ and $G^{-1}(0) = \inf\{y : G(y) > 0\}$, and note that G^{-1} may be discontinuous (but is left-continuous). Discontinuities require modifying almost all the arguments used in the proof of Theorem 1. As in Appendix 1, we relegate to the end of the section the proofs of all intermediate results.

Let $I_0 = (y_0^l, y_0^u)$ be a longest interval in $[0, G^{-1}(1)]$ to which G assigns measure 0; let $I_1 = (y_1^l, y_1^u)$ be a longest such interval disjoint from I_1, and so on. Then, every open interval of prizes that has measure zero is contained in one of the intervals $I_0, I_1, ...$. And for any $\varepsilon > 0$, there is a K such that the lengths of I_{K+1}, $I_{K+2}, ...$ sum up to less than ε.

The definitions of R_i^n, A_i^n, A^n, and T^n are as in Appendix 1. The definition of T, however, must be changed. First, by Helly's selection theorem, we take a converging subsequence of the sequence A^n; denote its limit by $A : B \to [0,1]$. This function is weakly increasing (because each A^n is). For the rest of the proof, denote this converging subsequence by A^n (with the corresponding sequence $T^n = (G^n)^{-1} \circ A^n$).

Let $T = G^{-1} \circ A$. Since G may not have full support, we now have that $T(0) = \inf\{z : G(z) > 0\}$ and $T(b_{\max}) = G^{-1}(1)$; in addition, T is still (weakly) increasing (compare to properties (1)-(3) from Appendix 1).

In addition, F^n converges pointwise to F, but $(G^n)^{-1}$ may not converge pointwise to G^{-1}. It is, however, easy to check that $\lim_n (G^n)^{-1}(r) = G^{-1}(r)$ unless r is the value of G on an interval $I_k = (y_k^l, y_k^u)$; moreover, $\lim_n (G^n)^{-1}(r) \geq G^{-1}(r)$ for every r that is the value of G on an interval $I_k = (y_k^l, y_k^u)$, but it can happen that $\lim_n (G^n)^{-1}(r) = y_k^u$ and $G^{-1}(r) = y_k^l$.

The discontinuities in G^{-1} imply that T may not be continuous, so Lemma 1 does not hold. Points of discontinuity, however, correspond to open intervals of prizes that have measure zero. More precisely, we have the following result.

Lemma 6 *For any $t > 0$ in B, one of the following conditions holds:*

1. *T is continuous at t, that is, for any sequences $q^m \uparrow t$ and $r^m \downarrow t$, we have $\lim T(q^m) = \lim T(r^m) = T(t)$.*

2. *There is some $k = 1, 2, \ldots$ such that for any sequences $q^m \uparrow t$ and $r^m \downarrow t$, we have $\lim T(q^m) = y_k^l$ and $\lim T(r^m) = y_k^u$. Moreover, $\lim A(q^m) = G(y_k^l)$ and $\lim A(r^m) = G(y_k^u)$.*

Using Lemma 6, we define another function T^* on B by setting $T^*(t) = \lim T(r)$ for some sequence $r \downarrow t$ (and $T^*(b_{\max}) = G^{-1}(1)$). The monotonicity of T guarantees that $T^*(t)$ is well-defined. In addition, it is easy to check that T^* is (weakly) increasing, right-continuous, and continuous at every bid t such that condition 1 from Lemma 6 holds. Note that T^* may not be an extension of T, because when $\lim T(r) \neq T(t)$, we have that $T^*(t) = \lim T(r) \neq T(t)$.

Consider now a bid $t > 0$ such that condition 2 from Lemma 6 holds. Denote this bid t by t_k, where k is described in condition 2. Then, there is a bid $t' < t_k$ such that $A(t') = A(t) = G\left(y_k^l\right)$, so A is constant on an interval below t_k. Indeed, if $A(t') < G\left(y_k^l\right)$ for all $t' < t_k$, then, as in the proof of Lemma 6, for large n no player would bid any t' slightly below t_k. This would be so, because bidding slightly above t_k would almost certainly give a prize no lower than y_k^u, whereas bidding t' would almost certainly give a prize no higher than y_k^l. Let

$$t_k^l = \inf\left\{t' : A(t') = G\left(y_k^l\right)\right\} < t_k.$$

It is also true that every maximal interval on which T^* is constant with a value lower than $G^{-1}(1)$ is $\left[t_k^l, t_k\right)$ for some k. Indeed, consider a maximal nontrivial interval with lower bound t^l and upper bound t^u on which the value of T^* is $y < G^{-1}(1)$. It suffices to show that $T^*(t^u) > y$, because then condition 2 from Lemma 6 applies to t^u, which implies that $t^u = t_k$ for some k; and the maximality of $\left[t^l, t^u\right)$ yields $t^l = t_k^l$. Suppose that $T^*(t^u) = y$. Then, for large enough n bidding t^u almost certainly gives a prize at most slightly higher than y, whereas bidding slightly above t^l almost certainly gives a prize not much lower than y. But then, for large enough n, no player bids in some neighborhood of t^u, because bidding slightly above t^l leads to a higher payoff. This contradicts the No-Gap Property, because $y < G^{-1}(1)$.

Because G^{-1} may be discontinuous, T^n need not converge uniformly to T or T^*, even on the set of points at which they are continuous. In particular, for a $t \in \left[t_k^l, t_k\right)$ it may be that $T^n(t) = (G^n)^{-1}(A^n(t)) \geq y_k^u$ for arbitrarily large n, whereas $T(t) = T^*(t) = y_k^l$. Nevertheless, T^n "converges uniformly" except on some neighborhoods of a finite number of intervals $[t_k^l, t_k]$. More precisely, we say that T^n *converges uniformly to T^* up to β* on a set C if there exists an N such that for every $n \geq N$ and $t \in C$ we have that

$$|T^n(t) - T^*(t)| < \beta.$$

We then have the following modification of Lemma 2.

Lemma 7 *For every $\beta > 0$, there exists a number K such that for every $\gamma > 0$, T^n converges uniformly to T^* up to β on the complement of*

$$O_\gamma = \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k + \gamma).$$

We now relate players' equilibrium behavior in large contests to the inverse tariff T^*. Define BR_x, $BR(\varepsilon)$, and $BR_x(\varepsilon)$ as in Appendix 1 with T^* instead of T (the maximal payoff is achieved because T^* is increasing and right-continuous, so is upper semi-continuous). Define *the mass expended (in the n-*th contest*) in an interval of bids I by players with type $x \in S$* as $\left(\sum_{i=1}^{n} \Pr(\sigma_i^n \in S \times I)\right)/n$. We then have the following result, which we use in proving the remaining results.

Lemma 8 *For all k and any $\varepsilon > 0$ and $L > 0$, there exists $\gamma > 0$ such that for sufficiently large n we have that:*

(i) The mass expended in $(t_k^l - \gamma, t_k^l + \gamma)$ by players with types x for which $t_k^l \notin BR_x(\varepsilon)$ is less than $\varepsilon/3L$;

(ii) The mass expended in $(t_k - \gamma, t_k]$ by players with types x for which $t_k \notin BR_x(\varepsilon)$ is less than $\varepsilon/3L$.

In addition, for any $\alpha > 0$, for sufficiently large n we have that:

(iii) The mass expended in $\left[t_k^l + \alpha, t_k - \alpha\right]$ by all players is less than $\varepsilon/3L$.

Lemma 3 must also to be modified.

Lemma 9 *For every $\varepsilon > 0$, there exist K such that for every $\gamma > 0$, there is an N such that for every $n \geq N$ in the equilibrium of the n-th contest every best response of every type x_i^n of every player i belongs to*

$$BR_{x_i^n}(\varepsilon) \cup \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k).$$

Strict single crossing no longer implies that BR_x is a singleton. Instead, we have the following result.

Lemma 10 *If strict single crossing holds, then for all but a countable number of types the set BR_x is a singleton. For those types for which it is not a singleton, BR_x contains precisely two elements: t_k^l and t_k for some k. The correspondence that assigns to type x the set BR_x is weakly increasing (i.e., for any $x' < x''$, if $t' \in BR_{x'}$ and $t'' \in BR_{x''}$, then $t' \leq t''$) and upper hemi-continuous.*

Let $br(x) = \min BR_x$, and note that br is increasing and left continuous, and is not right continuous precisely at types x for which BR_x is not a singleton. We then have the following corollary of Lemmas 8, 9, and 10, which is a modification of Corollary 1.

Corollary 2 *For every $\varepsilon > 0$, there is an N such that for $n \geq N$ at least a fraction $1 - \varepsilon$ of of players i bid in $(br(x_i^n) - \varepsilon, br(x_i^n) + \varepsilon)$ with probability at least $1 - \varepsilon$.*

To prove part (b) of the theorem it remains to show that $T^* \circ br$ is the assortative allocation. This is done by the following lemma, which is a modification of Lemma 5 that accommodates the discontinuities in T^* and br.

Lemma 11 *$G^{-1}(F(x)) = T^*(br(x))$ for any type $x > 0$.*

To complete the proof, it remains to show (a) in the statement of the theorem. To do so, we use the following result.

Lemma 12 *For every $\varepsilon, \delta > 0$, there is an N such that for $n \geq N$, each type x from a set whose F^n-measure is at least $1 - \varepsilon$ bids at least with probability $1 - \varepsilon$ a t, and obtains at least with probability $1 - \varepsilon$ a y such that for some r in BR_x,*

$$|t - r| < \delta \text{ and } |y - T^*(r)| < \delta.$$

To see that Lemma 12 implies (a) in the statement of the theorem, choose some $\varepsilon > 0$. Lemma 12 shows that for every $\delta > 0$, there is an N such that for $n \geq N$ and for a fraction $1 - \varepsilon$ of players i, the F_i^n-measure of their types x_i^n that satisfy the condition of Lemma 12 is at least $(1 - \varepsilon)$. This means that each such player i obtains with probability at least $1 - \varepsilon$ a prize y that differs by at most δ from the prize $T^*(r)$ for some optimal bid r of the player's type. For types $x_i^n > 0$ such that $br(x_i^n)$ is a unique optimal bid, this yields (a) by Lemma 11. However, by Lemma 9 and strict single crossing, there is only a countable number of other types x_i^n. And the F-measure of such types is 0 since F has no atoms, so the F^n-measure of such types is arbitrarily small for sufficiently large n.

2.1 Proof of Lemma 6

Let $\lim T(q^m) = y'$ and $\lim T(r^m) = y''$. Both limits y' and y'' exist and $y' \leq y''$ by monotonicity. Suppose that $y' < y''$. If G assigns a positive measure to (y', y''), then it assigns a positive measure to any interval with endpoints sufficiently close to y' and y''. In such a case, we obtain a contradiction by arguments similar to those used in the proof of Lemma 1. Indeed, for sufficiently large n no bidder would bid slightly below t, because bidding slightly above t would almost certainly give a better prize.

Thus, G assigns measure zero to (y', y''). This implies that $(y', y'') \subseteq (y_k^l, y_k^u)$ for some k. By definition, T takes values in $[0, y_k^l] \cup [y_k^u, 1]$, so $y' = y_k^l$ and $y'' = y_k^u$. Moreover, the monotonicity of T implies that k is the same for any sequences $q^m \uparrow t$ and $r^m \downarrow t$. It remains to show that $\lim A(q^m) = G(y_k^l)$ and $\lim A(r^m) = G(y_k^u)$.

For this, note that if $\lim A(q^m) > G(y_k^l)$, then $\lim T(q^m) > y_k^l$. Similarly, if $\lim A(r^m) > G(y_k^u)$, then $\lim T(r^m) > y_k^u$. The inequalities $\lim A(q^m) < G(y_k^l)$ and $\lim A(r^m) < G(y_k^u)$ can be ruled out similarly if G does not have atoms at y_k^l or y_k^u. Suppose that G has an atom at y_k^u and $\lim A(r^m) < G(y_k^u)$. Since $\lim T(r^m) = y_k^u$, $A(r^m) > G(y_k^l)$ for sufficiently large m. Take two numbers r^m such that $G(y_k^l) < A(r^m) < G(y_k^u)$; denote them by $t' < t''$. Then, for sufficiently large n any player obtains a prize close to y_k^u with arbitrarily high probability by bidding any $t \in [t', t'']$. Thus, for sufficiently large n, no player would bid in the interval $[(t' + t'')/2, t'']$ with positive probability. This contradicts the No-Gap Property.

Suppose that G has an atom at y_k^l and $\lim A(q^m) < G(y_k^l)$. Then, for sufficiently large n bidding q^m almost certainly gives a prize at most slightly better than y_k^l. In contrast, bidding r^m almost certainly gives a prize at least as good as y_k^u. This follows directly from (1) if G has an atom at y_k^u. If G does not, then this again follows from (1) for large enough n, because $A(r^m) > G(y_k^u)$ for any m. For large enough n a contradiction with the No-Gap Property is obtained similarly to the last part of the proof of Lemma 1 that deals with $U(0, y, t)$ strictly increasing in y.

2.2 Proof of Lemma 7

The proof is analogous to the proof of Lemma 2. Take a K such that the lengths of I_{K+1}, $I_{K+2}, \ldots$ sum up to less than $\beta/2$. Take any $\gamma > 0$, and suppose to the contrary that there is an increasing sequence of integers $n_1, n_2, \ldots, n_m, \ldots$ such that for every n_m there is some bid $t_m \notin O_\gamma$ with $|T^{n_m}(t_m) - T^*(t_m)| \geq \beta$. Passing to a subsequence if necessary, we assume that the sequence $t_m \to t$. Take q' and q'' such that $q' < t < q''$ and $T^*(q'') - T^*(q') < \beta/2$,[12] and

$$[q', q''] \subset B - \bigcup_{k=1}^{K} [t_k^l, t_k].$$

This is possible, since the lengths of I_{K+1}, $I_{K+2}, \ldots$ sum up to less than $\beta/2$. In addition, for large enough k we have that $|T^{n_k}(q') - T^*(q')| < \beta/2$ and $|T^{n_k}(q'') - T^*(q'')| < \beta/2$, since

[12] If $t = 0$, take $q' = 0$.

the length of each I_{K+1}, $I_{K+2}, ...$ is less than $\beta/2$. The rest of the proof coincides with the proof of Lemma 2.

2.3 Proof of Lemma 8

First, observe that the maximal payoff of type x, attained at any bid in BR_x, is still continuous in x. Indeed, upper semi-continuity of $T^*$is all that is needed for the continuity of the maximal payoff. This observation implies that there exists a $\delta > 0$ such that for any type x any bid in the complement of $BR_x(\varepsilon)$ gives type x a payoff lower by at least δ than any bid in BR_x.

For (i), suppose the contrary that for any $\gamma > 0$ there are arbitrarily large n such that the mass expended in $(t_k^l - \gamma, t_k^l + \gamma)$ by players with types x for which $t_k^l \notin BR_x(\varepsilon)$ is at least $\varepsilon/3L$. Take γ small enough so that the payoff that such players obtain by bidding slightly more than any bid in BR_x is higher by $\delta/2$ than the payoff that they would obtain by bidding $t_k^l - \gamma$ and getting y_k^l.[13]

Suppose first that $t_k^l > 0$. By monotonicity of A and the definition of t_k^l, we have that $A\left(t_k^l - \gamma\right) < G\left(y_k^l\right)$. Take a positive $\alpha < \varepsilon/6L$ such that $A\left(t_k^l - \gamma\right) < G\left(y_k^l\right) - \alpha$. For any $t \geq t_k^l - \gamma$ and sufficiently large n, if $A^n(t) < G\left(y_k^l\right) - \alpha/2$, then no player of type x such that $t_k^l \notin BR_x(\varepsilon)$ bids t, because by bidding t such a player would obtain with high probability a prize no higher than y_k^l, and therefore would obtain a higher payoff by bidding slightly more than any bid in BR_x.

Let γ_n be defined by $t_k^l - \gamma_n = \inf\left\{t : A^n(t) \geq G\left(y_k^l\right) - \alpha\right\}$. Since $A\left(t_k^l - \gamma\right) < G\left(y_k^l\right) - \alpha$ and $A^n(t)$ is right-continuous, we have that $\gamma_n < \gamma$ (for sufficiently large n). And since for every $t < t_k^l - \gamma_n$ we have $A^n(t) < G\left(y_k^l\right) - \alpha/2$ (by definition of γ_n), players with types x for which $t_k^l \notin BR_x(\varepsilon)$ must expend the mass of at least $\varepsilon/3L$ in $[t_k^l - \gamma_n, t_k^l + \gamma)$.

If more than half of this mass is expended in $\left(t_k^l - \gamma_n, t_k^l + \gamma\right)$, then we have that $A^n\left(t_k^l + \gamma\right) > A^n\left(t_k^l - \gamma_n\right) + \varepsilon/6L \geq G\left(y_k^l\right) - \alpha + \varepsilon/6L > G\left(y_k^l\right)$. This cannot happen

[13] To see why bidding slightly above any $t \in BR_x$ gives at least a payoff close to $U(x, T^*(t), t)$, consider the following two cases:

(a) $G^{-1}(A(r)) > T^*(t)$ for all $r > t$; in this case, since $\lim_n (G^n)^{-1}(A(r)) \geq G^{-1}(A(r))$, for any $r > t$, if n is sufficiently large, then $(G^n)^{-1}(A(r)) > T^*(t)$. This implies that a player obtains a prize higher than $T^*(t)$ with arbitrarily high probability by bidding r.

(b) $G^{-1}(A(r)) = T^*(t)$ for $r > t$ close enough to t; in this case, $T^*(r) = T^*(t)$ for all such r. This implies that $t = t_{k'}^l$ for some k'. The claim now follows from left-continuity of G^{-1} and the fact that $\lim_n (G^n)^{-1}(q) \geq G^{-1}(q)$ for any q.

for sufficiently large n, because for $t \in \left[t_k^l, t_k\right)$ is $A(t) = G\left(y_k^l\right)$. Thus, the players with types x for which $t_k^l \notin BR_x(\varepsilon)$ bid precisely $t_k^l - \gamma_n$ with probability at least $\varepsilon/6L$. Since these players tie with each other at $t_k^l - \gamma_n$, by bidding $t_k^l - \gamma_n$ they must obtain a prize of a specific type y with probability 1, even if they lose all ties at $t_k^l - \gamma_n$. (Otherwise, each of them could obtain a higher payoff by bidding slightly above $t_k^l - \gamma + \gamma_n$ and winning the ties at $t_k^l - \gamma_n$.) But a player who loses all ties at $t_k^l - \gamma_n$ has rank order no higher than $G\left(y_k^l\right) - \alpha$, by definition of γ_n, so $y \leq y_k^l$. Therefore, such a player would obtain a strictly higher payoff by bidding slightly more than any bid in BR_x.

Now suppose that $t_k^l = 0$. Then $A(t_k^l) \leq G\left(y_k^l\right)$. The case $A(t_k^l) < G\left(y_k^l\right)$ is handled as in the case $t_k^l > 0$ above. Suppose that $A(t_k^l) = G\left(y_k^l\right)$. Then, for any $\gamma > 0$ such that $t_k^l + \gamma < t_k$, for sufficiently large n the mass expended in $\left(t_k^l, t_k^l + \gamma\right)$ by all players is smaller than $\varepsilon/6L$, because $A(t) = G\left(y_k^l\right)$ for any $t \in \left(t_k^l + \gamma, t_k\right)$. Thus, if (i) does not hold, for sufficiently large n the mass expended precisely at t_k^l by the players with types x for which $t_k^l \notin BR_x(\varepsilon)$ is at least $\varepsilon/6L$, and so the ranking of a player who ties at t_k^l and loses is at most $G\left(y_k^l\right) - \varepsilon/12L$. But in this case each player of type x for which $t_k^l \notin BR_x(\varepsilon)$ would strictly prefer bidding slightly more than any bid in BR_x to bidding t_k^l, a contradiction.

To show (ii), note that if $t_k = t_{k'}^l$ for some k', then (ii) follows from (i). Thus, suppose that $t_k \neq t_{k'}^l$ for any k'. Suppose the contrary that for any $\gamma > 0$ there is an arbitrarily large n such that the mass expended in $(t_k - \gamma, t_k]$ by players with types x for which $t_k \notin BR_x(\varepsilon)$ is at least $\varepsilon/3L$. Take γ small enough so that the payoff that such players obtain by bidding slightly more than any bid in BR_x is higher by $\delta/2$ than the payoff that they would obtain by bidding $t_k - \gamma$ and getting y_k^u. Observe that for sufficiently large n, by bidding t_k any player almost certainly obtains a prize at most slightly better than $T^*(t_k) = y_k^u$. This is so, because $t_k \neq t_{k'}^l$ and so $A(t_k) \neq G(y_{k'}^l)$ for any k'. Therefore, for large enough n a player with type x for which $t_k \notin BR_x(\varepsilon)$ would be better off bidding slightly above any $t \in BR_x$ than bidding in $(t_k - \gamma, t_k]$.

Part (iii) follows immediately from the fact that the value of A on $\left[t_k^l, t_k\right)$ is $G\left(y_k^l\right)$, by the definition of t_k^l.

2.4 Proof of Lemma 9

Take a $\delta > 0$ such that for any type x any bid in the complement of $BR_x(\varepsilon)$ gives type x a payoff lower by at least δ than any bid in BR_x. Take $\beta > 0$ such that for any type x, bid t, and prizes y' and y'' with $|y' - y''| \leq \beta$ we have

$$|U(x, y', t) - U(x, y'', t)| \leq \frac{\delta}{3}.$$

Next, take a K guaranteed by Lemma 7 for this β. In addition, take K large enough so the lengths of I_{K+1}, $I_{K+2}, \ldots$ sum up to less than $\beta/2$. Finally, for any $\lambda > 0$ take an N_λ that satisfies the definition of uniform convergence up to β on the complement of O_λ. (Note that K is the same for all λ.)

Suppose to the contrary of the statement of the lemma that there is a $\gamma > 0$ and a subsequence of contests such that a type x_i^n of player i in the n-th contest has a best response t^n to the strategies of the other players that does not belong to $BR_{x_i^n}(\varepsilon) \cup \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k)$. As usual, we assume that the subsequence is the entire sequence; moreover, we assume that $x_i^n \to x^*$ and $t^n \to t^*$.

Consider the following two cases:

A. ($t^* \neq t_k$ for any $k = 1, ..., K$) In this case, for some $\lambda > 0$ there is a neighborhood of t^* that is disjoint from O_λ. By uniform convergence of T^n to T^* up to β on the complement of O_λ,

$$U\left(x_i^n, T^n\left(t^n\right), t^n\right) - U\left(x_i^n, T^*\left(t^n\right), t^n\right) \leq \frac{\delta}{3}$$

for $n \geq N_\lambda$. And because $t^n \notin BR_{x_i^n}(\varepsilon)$, for any $t \in BR_{x_i^n}$ we have

$$U\left(x_i^n, T^*\left(t\right), t\right) - U\left(x_i^n, T^*\left(t^n\right), t^n\right) \geq \delta.$$

Thus, we obtain

$$U\left(x_i^n, T^*\left(t\right), t\right) - U\left(x_i^n, T^n\left(t^n\right), t^n\right) \geq \frac{2\delta}{3}.$$

Observe that any bid t' higher than t guarantees, for sufficiently large n, a prize not much worse than $T^*(t)$ with arbitrarily high probability.[14]

We will now show that by bidding t^n, for sufficiently high n type x_i^n obtains with arbitrarily high probability a prize no better than $T^n\left(t^n\right) + \beta$. Indeed, since t^* does not belong to $\left[t_k^l, t_k\right]$ for any $k \leq K$, we have that $A\left(t'\right)$ is bounded away from $G\left(y_1^l\right), \ldots, G\left(y_K^l\right)$ for t' sufficiently close to t^*. Therefore, $A^n\left(t^n\right)$ is also bounded away from $G\left(y_1^l\right), \ldots, G\left(y_K^l\right)$ for sufficiently large n. And for sufficiently large n, bidding t^n gives with arbitrarily high probability a rank order arbitrarily close to $A^n\left(t^n\right)$. Since the lengths of I_{K+1}, $I_{K+2}, \ldots$ sum up to less than $\beta/2$, and for any r other than $G\left(y_1^l\right), \ldots, G\left(y_K^l\right)$ and sufficiently large n the difference between $(G^n)^{-1}(r)$ and $G^{-1}(r)$ is no larger than the length of I_{K+1}, by bidding t^n a player obtains with arbitrarily high probability a prize no better than $(G^n)^{-1}\left(A^n\left(t^n\right)\right) + \beta$.

Therefore, by definition of β, we have that by bidding t^n type x_i^n obtains a payoff that is higher than $U\left(x_i^n, T^n\left(t^n\right), t^n\right)$ by at most slightly more than $\delta/3$. Consequently, for sufficiently large n player i would obtain by bidding some $t' > t$ a payoff strictly higher than by bidding t^n, a contradiction.

[14] To see why, see the previous footnote.

B. ($t^* = t_k$ for some $k = 1, ..., K$) Then, consider a t^{**} slightly higher than t^*, such that t^{**} does not belong to $[t_k^l, t_k]$ for $k = 1, ..., K$, and such that: (i) for sufficiently large n the payoff (of any player) in the n-th contest of bidding t^{**} is not much lower than the payoff of bidding t^n; (ii) for sufficiently large n, we have that the difference between $U(x_i^n, T^*(t), t)$ for any t in $BR_{x_i^n}$ and $U(x_i^n, T^*(t^{**}), t^{**})$ is not much lower than δ. This latter condition is possible because, by definition, $(x^*, t^*) \notin BR(\varepsilon)$, and by right continuity of T^* at t^*. Now, using (ii), apply an argument analogous to that from case A with t^{**} playing the role of t^n, with a contradiction obtained by referring to (i).

2.5 Proof of Lemma 10

Monotonicity of the correspondence follows from strict single crossing, and upper hemi-continuity follows from standard arguments.[15]

Suppose that BR_x contains a pair of bids $t_1 < t_2$. Below we will show that for any $\varepsilon > 0$ and any interval $[a, b]$ such that $t_1 < a$ and $b < t_2$, for sufficiently large n the mass expended in $[a, b]$ by all players is at most ε. This implies that the function A, and therefore T^*, is constant on every such interval $[a, b]$, and therefore on (t_1, t_2). But $T^*(t_2) > T^*(t_1)$ because $t_1 < t_2$ are in BR_x, so by definition of the discontinuity points t_k of T^* we must have $(t_1, t_2) \subseteq (t_k^l, t_k)$ for some k. And because $BR_x \subseteq B \backslash \cup \bigcup_{k=1}^{\infty} (t_k^l, t_k)$, we have that $t_1 = t_k^l$ and $t_2 = t_k$.

It remains to show that for any $\varepsilon > 0$, for sufficiently large n the mass expended in $[a, b]$ by all players is at most ε. We will show this for $\varepsilon/2$ and players of types lower than x (a similar argument applies to types higher than x). Choose $x' < x$ such that $F(x) - F(x') < \varepsilon/3$. For sufficiently small $\lambda > 0$, $\sup \cup_{z \le x'} BR_z(\lambda) < a$. (This is because $x' < x$ and $t_1 \in BR_x$, so every bid in BR_z is at most $t_1 < a$.)

Therefore, by Lemma 9, there is some K such that for every $\gamma > 0$ and sufficiently large n any bid in $[a, b]$ made by a player of type $z \le x'$ in the n-th contest is in $\bigcup_{k=1}^{K} (t_k^l - \gamma, t_k)$. Consider one of these K intervals for which $(t_k^l - \gamma, t_k) \cap [a, b] \neq \varnothing$. Since $\sup \cup_{z \le x'} BR_z(\lambda) < a \le t_k$, t_k is not in $BR_z(\lambda)$ for any $z \le x'$. If $t_k^l > \sup \cup_{z \le x'} BR_z(\lambda)$, then by (i) of Lemma 8 there exists a γ such that for sufficiently large n the mass expended in $(t_k^l - \gamma, t_k)$ by players of type $z \le x'$ is less than $\varepsilon/6K$. If $t_k^l \le \sup \cup_{z \le x'} BR_z(\lambda)$, then by (ii) and (iii) of Lemma 8, for sufficiently large n the mass expended in $[a, t_k)$ by players of type $z \le x'$ is less than $\varepsilon/6K$.

[15] More precisely, this follows from the fact that BR_x is the set of all t such that $(t, T^*(t))$ maximizes type x's utility over the closure of the graph of T^*, which is a compact set.

Therefore, for large enough n the mass expended in $[a, b]$ by players of type $z \leq x'$ is smaller than $\varepsilon/6$, and because $F(x) - F(x') < \varepsilon/3$, the mass expended in $[a, b]$ by players of type $z \leq x$ is smaller than $\varepsilon/2$.

2.6 Proof of Corollary 2

Choose $\varepsilon > 0$. Lemma 10 implies that there is a finite number of intervals of types with total F-mass $\varepsilon/2$, such that for every type x not in one of these intervals, $BR_x \subseteq (br(x) - \varepsilon, br(x) + \varepsilon)$.[16] Consider the F-mass $1 - \varepsilon/2$ of types x with the last property, and let K be the one in the statement of Lemma 9. Then, by Lemma 9 and Lemma 8 for $L = K$, for sufficiently large n, at most an F-mass $\varepsilon/2$ of those types bid outside of $(br(x) - \varepsilon, br(x) + \varepsilon)$.

2.7 Proof of Lemma 11

The proof is analogous to that of Lemma 5. Consider an arbitrary type x. Define $x^{\min} = \min\{z : br(x) \in BR_z\}$ and $x^{\max} = \max\{z : br(x) \in BR_z\}$. By strict single crossing, BR_z has only one element $br(z) = br(x)$ for all $z \in (x^{\min}, x^{\max})$; it may have two elements for $z = x^{\min}$ or $x^{\max}$, in which cases $br(x)$ is the higher one and the lower one of the two, respectively.

The claim that $G^{-1}(F(x^{\min})) = G^{-1}(F(x^{\max}))$ is obtained by the same argument as in the proof of Lemma 5. The rest of the proof requires the following minor changes when $BR_{x^{\min}}$ has two elements (and analogous changes when $BR_{x^{\max}}$ has two elements):

1. Instead of $x^{\min}$, we consider $\underline{x}^{\min} = \min\{z : br(x^{\min}) \in BR_z\}$, and compare the equilibrium bids of every player with type lower than $\underline{x}^{\min} - \delta$ to $br(x^{\min})$, and the equilibrium bids of every player with type higher than $x^{\min}$ to $br(\underline{x}^{\min} - \delta)$. This change does not affect the arguments, since $G^{-1}(F(\underline{x}^{\min})) = G^{-1}(F(x^{\min}))$.

2. It may not be true that the equilibrium bids of every player with type lower than $\underline{x}^{\min} - \delta$ are lower than $br(x^{\min})$, or the equilibrium bids of every player with type higher than $x^{\min}$ are higher than $br(\underline{x}^{\min} - \delta)$, because players may bid in $\bigcup_{k=1}^{K}(t_k^l - \gamma, t_k) - BR(\varepsilon)$ (see Lemma 9). However, this happens only with vanishing probability as n grows large, so the arguments are again not affected.

[16] There is a $K > 0$ such that $\sum_{k>K}(t_k - t_k^l) < \varepsilon$. For each $k \leq K$ such that $BR_{x_k} = \{t_k^l, t_k\}$ for some type x_k, consider the interval of types $[x_k - \lambda, x_k + \lambda] \cap [0, 1]$, where λ is such that (continuous) F increases by no more than $\varepsilon/2K$ on any interval no larger than 2λ. The sum of the F-mass of these intervals is no larger than $\varepsilon/2$, and the sum of the "jumps" of br on the complement of these intervals is smaller than ε.

2.8 Proof of Lemma 12

Take any $\lambda > 0$. By Lemma 9, there is a large K such that for any $\gamma > 0$, if n is sufficiently large, the equilibrium bid of every player i in the n-th contest belongs with probability 1 to

$$BR_{x_i^n}(\lambda) \cup \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k).$$

Assume that K is, in addition, large enough so that the lengths of I_{K+1}, $I_{K+2}, \ldots$ sum up to less than $\delta/2$.

We first claim that for any $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, \ldots, K$, there exists an N_t such that for every $n \geq N_t$, a player who bids t in the n-th contest obtains (with high probability) a prize y such that $|y - T^*(t)| < \delta/2$. We will also show that there exists an $N = N_t$ that is common for all such bids t.

Suppose first that $t \neq t_k$ for any $k = 1, \ldots, K$. Since $A(t)$ differs from $G(y_k^l)$ and $G(y_k^u)$ for any $k = 1, \ldots, K$, any rank order close to $A(t)$ also differs from $G(y_k^l)$ and $G(y_k^u)$. By (1), for sufficiently large n, a player who bids t has (with high probability) a rank order close to $A(t)$; in particular, this rank order differs from $G(y_k^l)$ and $G(y_k^u)$. By the assumption that the lengths of I_{K+1}, $I_{K+2}, \ldots$ sum up to less than $\delta/2$, this implies that the difference between $T^*(t)$ and the prize obtained by a player who bids t is lower than $\delta/2$ (with high probability).

Suppose that $t = t_k$ for some $k = 1, \ldots, K$. By an argument analogous to the one used in the previous case, the prize obtained by a player who bids t cannot, as n increases, exceed $T^*(t)$ by $\delta/2$ with a probability that is bounded away from 0. And $T^*(t)$ cannot exceed this prize by $\delta/2$ with a probability that is bounded away from 0 as n increases, because the player would profitably deviate by bidding slightly above t, which would guarantee a prize no worse than $T^*(t)$ with arbitrarily high probability.

Now, note that the number N_t that was chosen for any bid t has the required property also for all bids close enough to t; in the case of $t = t_k$ for some $k = 1, \ldots, K$, we mean bids close enough and higher than t. That is, for every t there is a neighborhood W_t of that t with N_t that is common for all bids from this neighborhood. The family of sets W_t is an open covering of the compact set of bids t that satisfy $t \notin (t_k^l - \gamma, t_k)$ for $k = 1, \ldots, K$. Thus, it contains a finite subcovering, and any number N that exceeds numbers N_t for all elements of this finite subcovering has the required property.

This yields the lemma for bids $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, \ldots, K$. Indeed, note that BR_x is a singleton, and $br(x)$ is its only element, for all except a countable number of types x. Since F has no atoms, the set of such types has F-measure 1. And for such types x, equilibrium bids $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, \ldots, K$ belong to $(br(x) - \lambda, br(x) + \lambda)$. If λ

is sufficiently small, and x is bounded away from 0, then $|t-r| < \delta$ for $r = br(x)$, and $T^*(t) - T^*(r) \leq \delta/2$.[17] And if λ is sufficiently small, then also $T^*(r) - T^*(t) \leq \delta/2$, because $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, ..., K$ and the lengths of I_{K+1}, $I_{K+2}, ...$ sum up to less than $\delta/2$. Finally, by our first claim, the prize y obtained by bidding t must satisfy $|y - T^*(t)| < \delta/2$, so $|y - T^*(r)| < \delta$.

Now consider bids t such that t is in $(t_k^l - \gamma, t_k)$ for some $k = 1, ..., K$. By (iii) of Lemma 8, we can disregard bids t in $[t_k^l + \gamma, t_k - \gamma]$. Suppose that t is in $(t_k - \gamma, t_k)$ and $t_k \neq t_{k'}^l$ for all other $k' = 1, ..., K$. By (ii) of Lemma 8, one can assume that $t_k \in BR_{x_i^n}$.[18] We will show that for sufficiently small γ and for sufficiently large n, player i obtains by bidding t (with arbitrarily high probability) a prize y in $(T^*(t_k) - \delta, T^*(t_k) + \delta)$. First, note that player i cannot obtain by bidding t a prize lower than $T^*(t_k) - \delta$ (with probability bounded away from 0), because for small enough γ it would be profitable to deviate to bidding slightly above t_k, and obtain a prize not much lower than $T^*(t_k)$ with high probability. Player i cannot obtain by bidding t a prize higher than $T^*(t_k) + \delta$ (with probability bounded away from 0), because by (1), for any $r > t_k$ and sufficiently large n the rank order of player i is with arbitrarily high probability bounded above by $A(r)$. Thus, the upper bound on the prize follows from the assumption that $t_k \neq t_{k'}^l$ for all other $k' = 1, ..., K$, and the lengths of $I_{K+1}, I_{K+2}, ...$ sum up to less than $\delta/2$.

Finally, suppose that t is in $\left(t_k^l - \gamma, t_k^l + \gamma\right)$ for some $k = 1, ..., K$. By (i) of Lemma 8, one can assume that $t_k^l \in BR_{x_i^n}$. We will show that for sufficiently small γ and for sufficiently large n, equilibrium bidding in $\left(t_k^l - \gamma, t_k^l + \gamma\right)$ leads (with arbitrarily high probability) to a prize $y \in (T^*(t_k^l) - \delta, T^*(t_k^l) + \delta)$, except a small probability event. Indeed, by an argument similar to that from the previous case, such a bid cannot lead to a prize lower than $T^*(t_k^l) - \delta$ (with probability bounded away from 0). To obtain a prize higher than $T^*(t_k^l) + \delta$ with a nonvanishing probability, a player's expected rank order when bidding t cannot be lower than $G\left(y_k^l\right)$ by a nonvanishing constant. But, if a nonvanishing fraction of players win a prize higher than $T^*(t_k^l) + \delta$ with a nonvanishing probability by bidding in $\left(t_k^l - \gamma, t_k^l + \gamma\right)$, then the increase in expected rank order on the interval $\left(t_k^l - \gamma, t_k^l + \gamma\right)$ is bounded away

[17] Indeed, for types bounded away from 0, and for sufficiently small λ, we have that $U(x, y, t) > U(x, y', t')$ whenever $y - y' > \delta/2$ and $t - t' < \lambda$. (The assumption that types are bounded away fro 0 is essential, because we did not assume that $U(0, y, t)$ strictly increases in y.) However, since $r = br(x)$, we cannot have $U(x, T^*(t), t) > U(x, T^*(r), r)$.

[18] The lemma says only that the mass expended in $(t_k - \gamma, t_k]$ by types x for which $t_k \notin BR_x(\lambda)$ for some small $\lambda > 0$ is small. However, if $\lambda > 0$ is sufficiently small, then the mass of types x such that $t_k \notin BR_x$ but $t_k \in BR_x(\lambda)$ is small.

from 0 for all n, which contradicts the fact that $A^n(t_k^l + \gamma)$ approaches $G(y_k^l)$ as n increases.